

# Symmetry classification of scalar Ito equations with multiplicative noise

Giuseppe Gaeta<sup>\*†</sup> & Francesco Spadaro<sup>†‡</sup>

9/2/2020

## Abstract

We provide a symmetry classification of scalar stochastic equations with multiplicative noise. These equations can be integrated by means of the Kozlov procedure, by passing to symmetry adapted variables.

## 1 Introduction

In a recent paper [1] we have explicitly integrated the *stochastic logistic equation with multiplicative noise*

$$dx = (Ax - Bx^2) dt + \mu x dw, \quad (1)$$

where  $A, B, \mu$  are positive real constants. The motivation for this was provided by questions in Mathematical Biology, in particular Population Dynamics, where the logistic equation plays a central role. In this context, multiplicative noise is also known as *environmental noise*, as it models fluctuations due to changes in the environmental conditions and thus acting in the same way – and in a fully correlated manner – on all the individuals [2, 3]. (Uncorrelated fluctuations for different individuals give raise to noise terms proportional to  $\sqrt{x}$ ; in that context, one refers to this as *demographic noise* [2].)

In order to integrate (1), we have employed tools from the recently developed theory of *symmetry of stochastic differential equations* [4–18]. In particular, it was found that (1) admits a simple Lie symmetry and hence, thanks to a general constructive theorem by Kozlov [7–9] (see also [11–13, 17]), one can pass to symmetry-adapted variables allowing for a direct integration. See [1] for details.

---

<sup>\*</sup>Dipartimento di Matematica, Università degli Studi di Milano, via Saldini 20133 Milano (Italy) and SMRI, 00058 Santa Marinella (Italy); e-mail: [giuseppe.gaeta@unimi.it](mailto:giuseppe.gaeta@unimi.it)

<sup>†</sup>EPFL-SB-MATHAA-CSFT, Batiment MA - Station 8, CH-1015 Lausanne (Switzerland); e-mail: [francesco.spadaro@epfl.ch](mailto:francesco.spadaro@epfl.ch)

<sup>‡</sup>GNFM-INdAM

It is natural to wonder if the integrability of (1) was a lucky accident or, as it indeed appears more probable, (1) is one of a more and less wide family of integrable equations.

In the present work, we want to consider scalar Ito equations (we always assume the drift  $f(x, t)$  and noise  $\sigma(x, t)$  are smooth functions of their arguments)

$$dx = f(x, t) dt + \sigma(x, t) dw \quad (2)$$

with a noise term  $\sigma(x, t)$  corresponding to multiplicative noise, i.e.

$$\sigma(x, t) = S(t) x ;$$

we assume  $S(t) \neq 0$ , or we would have a deterministic equation. For equations in this class, we want to classify those admitting a simple Lie-point symmetry and which hence can be integrated by means of the Kozlov approach.

Thus the general form of equations to be considered is

$$dx = f(x, t) dt + S(t) x dw . \quad (3)$$

We will look for the most general symmetries allowed by our classification [10], i.e. W-symmetries; we will however restrict our attention to *simple* W-symmetries, as these are the only ones which can be used to integrate the equation [18]. That is, we look for symmetry vector fields of the form

$$X = \varphi(x, t, w) \partial_x + R w \partial_w , \quad (4)$$

where  $R$  is a real constant.

In this case the determining equations read [18]

$$\varphi_t + f \varphi_x - \varphi f_x + \frac{1}{2} \Delta(\varphi) = 0 \quad (5)$$

$$\varphi_w + \sigma \varphi_x - \varphi \sigma_x = R \sigma . \quad (6)$$

Here and below  $\Delta$  is the Ito Laplacian, which in our scalar case reads simply

$$\Delta(\Psi) := \frac{\partial^2 \Psi}{\partial w^2} + 2 \sigma \frac{\partial^2 \Psi}{\partial x \partial w} + \sigma^2 \frac{\partial^2 \Psi}{\partial x^2} . \quad (7)$$

In our discussion, we will assume the reader to be familiar with the standard theory of symmetry for deterministic differential equations [19–24] and with stochastic differential equations [25–31].

**Remark 1.** We recall that, as shown by Kozlov [7–9] (see also [10, 18] for the extension of Kozlov theorem to W-symmetries), once we have determined a simple symmetry of an Ito SDE, this is explicitly integrated by passing to the (symmetry adapted) new variable

$$y = \int \frac{1}{\varphi(x, t, w)} dx ; \quad (8)$$

thus our classification of equations (with multiplicative noise) admitting a simple symmetry also provides a classification of equations (in this class) which can be explicitly integrated by the Kozlov change of variables.  $\odot$

**Remark 2.** The paper [8] also provides a symmetry classification of scalar SDEs; this refers to *standard deterministic* symmetries, while the present one deals with the (more general) framework of *W-symmetries*, including also the case of *standard random* symmetries [10]. On the other hand, in [8] one considers general scalar Ito equations, while we only deal with a specific form of the diffusion coefficient  $\sigma(x, t)$ .  $\odot$

## 2 The basic classification

The determining equations for our specific form of Ito equation with multiplicative noise (3) restrict the scenario – in terms of possible drifts term – for which symmetries are present: only when the drift  $f(x, t)$  has a specific form, the deterministic equations allow solutions  $\varphi$ , i.e. symmetries (standard or W). At this point we do not want to discuss about regularity requirements of the functions involved, but we will rather derive the constrained form of  $f(x, t)$ . Thus, from now on, we will assume all functions we meet to be regular enough. In the following computation we will get the aforementioned constraints on  $f(x, t)$ , summarized in Proposition 1, by solving the determining equations by separation of variables.

For the specific form of (3), the second determining equation (35) reads

$$\varphi_w + S (x \varphi_x - \varphi - Rx) = 0 . \quad (9)$$

The general solution of this is<sup>1</sup>

$$\varphi(x, t, w) = x [R \log(x) + q(t, z)] , \quad (10)$$

where we have written

$$z := w - \frac{\log(x)}{S(t)} . \quad (11)$$

Restricting (34) to functions of this type, i.e. plugging (10) into (34), we get an equation of the form

$$x q_t + \chi_0 + \chi_1 q + \chi_2 q_z = 0 , \quad (12)$$

where we have written

$$\begin{aligned} \chi_0 &= \frac{R}{2} [2f[1 + \log(x)] + x (S^2 - 2 \log(x) f_x)] , \\ \chi_1 &= (f - x f_x) , \\ \chi_2 &= \frac{1}{2} x S + \frac{x S'}{S^2} \log(x) - \frac{f}{S} . \end{aligned}$$

---

<sup>1</sup>Note that by  $\log(x)$  we will always mean the natural logarithm; moreover it will be understood that we write  $\log(x)$  for  $\log(|x|)$ . This is specially justified when we look at equations like (1), with  $x$  describing a population – which is by definition non-negative – which is the motivation of our work.

Here both the functions  $f(x, t)$  and  $q(t, z)$  are unknown. Moreover, now our set of variables is  $(x, t, z)$ .

Differentiating (12) three times in  $x$  and once in  $z$ , eliminating a common factor  $(x^2 S(t))^{-1}$  (as stressed above  $S(t) \neq 0$ ), and writing

$$g(x, t) := f_{xxx}(x, t) \quad (13)$$

for ease of notation, we get

$$\begin{aligned} q_{zz}(t, z) [S'(t) + x^2 S(t) g(x, t)] + \\ + q_z(t, z) [2x^2 S^2(t) g(x, t) + x^3 S^2(t) g_x(x, t)] = 0. \end{aligned} \quad (14)$$

Now we can separate the  $x$  and  $z$  variables, which yields

$$-\frac{q_{zz}}{q_z} = \frac{2x^2 S^2(t) g(x, t) + x^3 S^2(t) g_x(x, t)}{S'(t) + x^2 S(t) g(x, t)}. \quad (15)$$

Here the l.h.s. is a function of  $z$  and  $t$ , while the r.h.s. is a function of  $x$  and  $t$ . Thus this relation can hold only if both sides are a function of  $t$  alone, i.e. if the l.h.s. is independent of  $z$  and the r.h.s. is independent of  $x$ .

Requiring that the l.h.s. of (15) is independent of  $z$  amounts to solving

$$q_{zz}^2 = q_z q_{zzz}; \quad (16)$$

this yields immediately (here  $Q_i(t)$  are arbitrary smooth functions)

$$q(t, z) = \exp[z Q_1(t)] Q_2(t) + Q_3(t). \quad (17)$$

**Remark 3.** Note that we can exclude the case  $Q_1(t) = 0$ . In fact, if this was the case we would have  $q(t, z) = Q_2(t) + Q_3(t)$ , but as these are both arbitrary functions, it is the same as assuming  $q(t, z) = Q_3(t)$ , which is obtained for  $Q_2(t) = 0$ .  $\odot$

As for the request that the r.h.s. of (15) is independent of  $x$ , this leads to

$$4S'g + x^3 Sg g_x + 5xS'g_x - x^4 Sg_x^2 + x^4 Sg g_{xx} + x^2 S'g_{xx} = 0. \quad (18)$$

This equation can be solved, yielding the general solution

$$g(x, t) = \frac{1}{x^2 S(t)} \left[ x^{\alpha(t)} \eta(t) - S'(t) \right], \quad (19)$$

where again  $\alpha$  and  $\eta$  are arbitrary smooth functions.

**Remark 4.** We pause a moment to note that in order to determine  $f(x, t)$  from a  $g(x, t)$  of this general form, we should solve (13) as an equation for  $f$ . In doing this, we should pay attention to the arbitrary function  $\alpha(t)$ : it turns out that if  $\alpha(t)$  is actually constant and takes either of the values  $\{-1, 0, +1\}$

we have special cases (one of the three integrations in  $x$  concerns a factor  $1/x$  and hence produces a logarithm), as discussed below.  $\odot$

We can now go back to (15); with the expression obtained above for  $q(t, z)$  and  $g(x, t)$ , this reads

$$Q_1(t) = -\alpha(t) S(t) . \quad (20)$$

Thus we set  $Q_1(t)$  to be as in (20), and with this we get

$$q(t, z) = \exp[-\alpha(t) S(t) z] Q_2(t) + Q_3(t) . \quad (21)$$

**Remark 5.** It may be noted that in this way we constrain the expression for (the possible)  $\varphi$ ; in particular, we have

$$\varphi(x, t) = x \left[ R \log(x) + x^{\alpha(t)} \exp[-\alpha(t) S(t) w] Q_2(t) + Q_3(t) \right] . \quad (22)$$

Thus  $\varphi$  can indeed depend on  $w$  (unless  $\alpha(t)$  or  $Q_2(t)$  are zero). Note also that (at least at this stage) we can have proper W-symmetries, i.e.  $R \neq 0$ ; moreover the possible dependence on  $\log(x)$  is related to  $R \neq 0$  – thus standard symmetries will not have any dependence on  $\log(x)$ .  $\odot$

Let us now go back to considering  $f(x, t)$ , i.e. the possible form of the symmetric Ito equations. Now that we have an expression for  $g(x, t)$  the form of  $f(x, t)$  is obtained by integrating it, i.e. solving (13) as an equation for  $f$ .

For the generic case, which in this context means  $\alpha(t) \neq -1, 0, +1$  (these cases should be dealt with separately), this yields

$$f(x, t) = [F_1(t) + x F_2(t) + x^2 F_3(t)] + \Psi(x, t) \quad (23)$$

where we have defined

$$\Psi(x, t) := \left[ (\log(x) - 1) x \frac{S'(t)}{S(t)} - \Phi(x, t) \frac{\eta(t)}{S(t)} \right] , \quad (24)$$

$$\Phi(x, t) := \frac{x^{1+\alpha(t)}}{[1 - \alpha(t)] \alpha(t) [1 + \alpha(t)]} . \quad (25)$$

As mentioned above, this expression holds provided  $\alpha(t)$  is not constant and equal to either 0 or  $\pm 1$ . In these special cases, the expression (23) still holds<sup>2</sup>, but now  $\Phi(x, t)$  should be defined in a different way:

$$\alpha(t) = -1 \quad \Phi(x, t) \rightarrow \Phi_{(-)}(x, t) := -\frac{1}{2} \log(x) , \quad (26)$$

$$\alpha(t) = 0 \quad \Phi(x, t) \rightarrow \Phi_{(0)}(x, t) := -x [1 - \log(x)] , \quad (27)$$

$$\alpha(t) = +1 \quad \Phi(x, t) \rightarrow \Phi_{(+)}(x, t) := -\frac{1}{4} x^2 [2 \log(x) - 3] . \quad (28)$$

---

<sup>2</sup>Indeed the term in square brackets in the r.h.s. of (23) correspond to the constants of integration – which of course depend on  $t$  – in the three  $x$  integrations.

We can insert these expressions in that for  $f(x, t)$ ; one should recall here that the  $F_k(t)$  are generic functions, so that adding to them any other function of time will not change their nature but just change them from  $F_k(t)$  to some other function  $\widehat{F}_k(t)$ , which we will still write as  $F_k(t)$ .

In this way, and writing for ease of notation

$$\Sigma(t) := \frac{S'(t)}{S(t)}, \quad \mathcal{F}(x, t) = F_1(t) + x F_2(t) + x^2 F_3(t),$$

we have

$$f(x, t) = \begin{cases} \mathcal{F}(x, t) + G(t) x^{1+\alpha(t)} + \Sigma(t) x \log x & (\alpha(t) \text{ generic}) \\ \mathcal{F}(x, t) + \Sigma(t) x \log x + H(t) x^2 \log x & (\alpha(t) = +1) \\ \mathcal{F}(x, t) + H(t) x \log x & (\alpha(t) = 0) \\ [F_1(t) + x F_2(t) + x^2 F_3(t)] + H(t) \log x & (\alpha(t) = -1) \end{cases} \quad (29)$$

Here  $G(t)$  and  $H(t)$  are arbitrary functions.

Before going on to discuss the different cases, we note that our discussion so far already identifies the – rather restricted – general class of equations of the form (3) which *could* admit some symmetry. We summarize the result of our discussion in this respect in the following Proposition 1; note that in there we have used the presence of arbitrary functions, and also write

$$\mathcal{H}(x, t) = H_1(t) + x H_2(t) + x^2 H_3(t),$$

in order to further simplify the writing of our functional forms.

**Proposition 1.** *Scalar SDEs of the form (3) may admit a symmetry only if  $f(x, t)$  is of the general form*

$$f(x, t) = \mathcal{F}(x, t) + G(t) x^{1+\alpha(t)} + \mathcal{H}(x, t) \log x. \quad (30)$$

**Remark 6.** It may be worth mentioning that this is a general form and provides necessary, but by no means sufficient, conditions for the existence of a symmetry; under closer scrutiny we will find out that actually this is too general to provide also sufficient conditions: e.g., a symmetry will be present only if  $H_3(t) = 0$ .  $\odot$

Let us come back to our task of classifying symmetric equations and symmetries. We should now insert these expressions for  $q(t, z)$  and  $f(x, t)$  in (34). We obtain quite complicate expressions – which we do not report here – in the generic and in the three special cases.

The relevant point is that all dependencies on  $x$  – including those in  $\log(x)$  – and  $z$  are now explicit, so that coefficients of different monomials in  $\{z, x, \log(x)\}$  should vanish separately, and the equation splits in several simpler ones.

**Remark 7.** It should also be noted that in our equations we have some factors  $x^{\alpha(t)}$ , see (19); if  $\alpha(t)$  is actually a constant and in particular if  $\alpha(t)$  is an

integer, this can interact with other terms. This can actually happen only for  $\alpha(t) = 0, \pm 1$ , see (30), and we have already warned that these should be treated separately (exactly for this reason); we will thus refer to the case where  $\alpha(t) \neq 0, \pm 1$  as the *generic* case, and to those where  $\alpha(t) = 0, \pm 1$  as *special* cases.  $\odot$

**Remark 8.** Note also that in the special cases we can always set  $\eta(t) = 0$  with no loss of generality.  $\odot$

### 3 Results

We will give here our results; the proof, which consists largely of a detailed computation, is confined to the “supplementary material” (see below).

In order to state our results more compactly, it will be convenient to introduce the notations

$$\Gamma_{\pm}(t) := \frac{S^2(t)}{2} \pm \frac{S(t)}{k} [\Sigma(t) - \chi(t)] ; \quad \chi(t) := \frac{Q'(t)}{Q(t)} . \quad (31)$$

**Theorem 1.** *The equations of the form (3) admit standard ( $R = 0$ ) or  $W$ -symmetries ( $R \neq 0$ ) for  $f(x, t)$  given by one of the expressions in the following table, where  $\mathcal{F}$  and  $\mathcal{G}$  and  $\theta$  are arbitrary functions (possibly zero):*

case	$S(t)$	$R$	$f(x, t)$
(a)	$S(t)$	0	$\Gamma_+(t)x + \Sigma(t)x \log x + \mathcal{G}(t)x^{1+k/S(t)}$
(b)	$S(t)$	0	$\Gamma_+(t)x + \Sigma(t)x \log x$
(c)	$S(t)$	0	$\mathcal{F}(t)x + \Sigma(t)x \log x$
(d)	$S(t)$	0	$\mathcal{F}(t)x + \theta(t)x \log x$
(e)	$S(t)$	$R$	$-\Gamma_-(t)x + \Sigma(t)x \log x$
(f)	$s_0$	0	$\mathcal{F}(t)x^2 + [s_0^2/2 - \chi(t)]x$
(g)	$s_0$	0	$\mathcal{F}(t) + [s_0^2/2 - \chi(t)]x$
(h)	$s_0$	$R$	$[s_0^2/2 - \chi(t)]x$

Their (standard or  $W$ ) symmetries are given by (4) with  $R$  as in the previous table and coefficient  $\varphi$  given by the following table, where  $k \neq 0$  and  $K$  (possibly  $K = 0$ ) are arbitrary constants:

case	$R$	$\varphi(x, t, w)$
(a)	0	$Q(t)e^{-kw}x^{1+k/S(t)}$
(b)	0	$KS(t) + Q(t)e^{-kw}x^{1+k/S(t)}$
(c)	0	$KS(t)x$
(d)	0	$\theta(t)x$
(e)	$R$	$R[\mathcal{F}(t)e^{-kw}x^{1+k/S(t)} + \theta(t)x + x \log x]$
(f)	0	$KS_0x + Q(t)e^{-s_0w}x^2$
(g)	0	$Kx + Q(t)e^{s_0w}$
(h)	$R$	$R[\theta(t)x + J(t)e^{s_0w}x^2 + x \log x]$

Furthermore, no other scalar Ito stochastic differential equations (3) beside those detailed here admit standard or  $W$ -symmetries.

**Proof.** As mentioned before, the proof consists in explicitly computing and carrying out, starting from Proposition 1 and the determining equations, the possible symmetries listed in the table above. Checking that these equations admit the given symmetries just requires direct elementary computations, i.e. checking that the determining equations (34), (35) are satisfied. The second part of the statement, i.e. that these are the only equations admitting symmetries, requires more involved computations. For a sake of readability, these computations are given explicitly in the supplementary material.  $\triangle$

## 4 Discussion and Conclusions

We have considered in full generality equations of the form

$$dx = f(x, t) dt + S(t) x dw ,$$

i.e. scalar stochastic differential equations with multiplicative noise, also known in population dynamics as equations with environmental noise.

We have investigated in which cases, i.e. for which functions  $f(x, t)$  and  $S(t)$ , they admit a symmetry (or more precisely a one-parameter group of symmetries), setting this search within the most general class of symmetries, i.e.  $W$ -symmetries.

Our detailed analysis had to consider a generic case and three special cases, depending on the function  $\alpha(t)$  introduced in (19); the special cases correspond to  $\alpha(t)$  being a constant function with value  $\{-1, 0, +1\}$ .

We have found that the most general functional form of  $f(x, t)$  admitting a symmetry (including the special cases) is

$$f(x, t) = \mathcal{F}(t) x + \mathcal{G}(t) x^{1+\alpha(t)} + \mathcal{H}(t) x \log(x) . \quad (32)$$

As for the symmetries, these are characterized by  $R$  and  $\phi$ , see (4); the most general form of  $\varphi$  turned out to be

$$\varphi(x, t, w) = \mathcal{A}(t) x + \mathcal{B}(t) \exp[kw] x^{1+\alpha(t)} + R x \log x . \quad (33)$$

The reader is referred to Theorem 1 for the specific form taken by the functions  $\{\mathcal{F}(t), \mathcal{G}(t), \mathcal{H}(t); \mathcal{A}(t), \mathcal{B}(t)\}$  as well as for  $\alpha(t)$  and  $S(t)$ , which are in many cases constrained; in particular, several of the specific cases obtained above require that  $S(t)$  is constant. Note also that in all subcases admitting such terms but one, we had  $\mathcal{H}(t) = [S'(t)/S(t)]$ ; this also means that the terms  $x \log x$  can be allowed in  $f(x, t)$  only if  $S(t)$  is actually constant.

It should be stressed that our work points out at a substantial difference with respect to the deterministic case: in fact, in that case the functional form of deterministic equations admitting symmetries (and of the symmetries) is largely

unconstrained, and actually for the closet analogue to the equations considered here, i.e. for dynamical systems, it is known that we always have symmetries (albeit these can not be determined algorithmically). We have found that the situation is completely different for stochastic equations: not only symmetry is a non-generic property, but the functional form of stochastic equations admitting symmetries is severely constrained. This is maybe the most relevant – and qualitative – result of our classification; unfortunately it also means that, at least for scalar equations, the value of symmetry methods is restrained to a rather specific class of equations.

Our classification would naturally call for further work:

- It follows from the general results of Kozlov theory [7–9] (in some cases, extended to include  $W$ -symmetries as well) that equations of the form (32) – in some cases with extra conditions on  $S(t)$  – can be integrated by passing to symmetry adapted variables. Thus each of our symmetric equations listed in Theorem 1 could be integrated by using the corresponding symmetries.
- An alternative – albeit strongly related – approach to integration of stochastic equations is that based on stochastic invariants [14–16]; it would be interesting to have a similar classification for invariants of scalar stochastic equations with environmental noise.
- We have classified equations with *environmental* noise; it would be natural to consider the same task for equations with *demographic* noise, or equations with both environmental and demographic noise (the so called *complete models* [2]).

These aspects lay outside our present scope and were not discussed here; we hope to tackle them in future work.

## References

- [1] G. Gaeta, “Integration of the stochastic logistic equation via symmetry analysis”, *J. Nonlin. Math. Phys.* **26** (2019), 454-467
- [2] O. Ovaskainen and B. Meerson, “Stochastic models of population extinction”, *Trends in Ecology and Evolution* **25** (2010), 643-652; *arXiv:1008.1162*
- [3] B. Spagnolo, D. Valenti and A. Fiasconaro, “Noise in Ecosystems: a short review”, *Math. Biosci. Eng.* **1** (2004), 185-211; *arXiv:q-bio/0403004*
- [4] G. Gaeta and N. Rodríguez-Quintero, “Lie-point symmetries and stochastic differential equations”, *J. Phys. A* **32** (1999), 8485-8505
- [5] G. Gaeta, “Lie-point symmetries and stochastic differential equations II”, *J. Phys. A* **33** (2000), 4883-4902
- [6] G. Unal, “Symmetries of Ito and Stratonovich dynamical systems and their conserved quantities”, *Nonlin. Dyn.* **32** (2003), 417-426
- [7] R. Kozlov, “Symmetry of systems of stochastic differential equations with diffusion matrices of full rank”, *J. Phys. A* **43** (2010), 245201
- [8] R. Kozlov, “The group classification of a scalar stochastic differential equations”, *J. Phys. A* **43** (2010), 055202
- [9] R. Kozlov, “On maximal Lie point symmetry groups admitted by scalar stochastic differential equations”, *J. Phys. A* **44** (2011), 205202
- [10] G. Gaeta and F. Spadaro, “Random Lie-point symmetries of stochastic differential equations”, *J. Math. Phys.* **58** (2017), 053503 [Erratum, *J. Math. Phys.* **58** (2017), 129901]
- [11] G. Gaeta, “Symmetry of stochastic non-variational differential equations”, *Phys. Rep.* **686** (2017), 1-62 [Erratum: **713**, 18]
- [12] G. Gaeta and C. Lunini, “On Lie-point symmetries for Ito stochastic differential equations”, *J. Nonlin. Math. Phys.* **24-S1** (2017), 90-102
- [13] G. Gaeta and C. Lunini, “Symmetry and integrability for stochastic differential equations”, *J. Nonlin. Math. Phys.* **25** (2018), 262-289
- [14] R. Kozlov, “Random Lie symmetries of Ito stochastic differential equations”, *J. Phys. A* **51** (2018), 305203
- [15] R. Kozlov, “Lie-point symmetries of Stratonovich stochastic differential equations”, *J. Phys. A* **51** (2018), 505201
- [16] R. Kozlov, “Symmetries of Ito stochastic differential equations and their applications”, in *Nonlinear systems and their remarkable mathematical structures*, N. Euler ed., CRC Press 2018

- [17] G. Gaeta, C. Lunini and F. Spadaro, “Recent advances in symmetry of stochastic differential equations”, *Rend. Mat. Appl.* **39** (2018), 293-306
- [18] G. Gaeta, “W-Symmetries of Ito stochastic differential equations”, *J. Math. Phys.* **60** (2019), 053501
- [19] D.V. Alexseevsky, A.M. Vinogradov and V.V. Lychagin, *Basic Ideas and Concepts of Differential Geometry*, Springer 1991
- [20] G. Cicogna and G. Gaeta, *Symmetry and Perturbation Theory in Nonlinear Dynamics*, Springer 1999
- [21] I.S. Krasil’schik and A.M. Vinogradov, *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*, A.M.S. 1999
- [22] P.J. Olver, *Application of Lie Groups to Differential Equations*, Springer 1986
- [23] P.J. Olver, *Equivalence, Invariants and Symmetry*, Cambridge University Press 1995
- [24] H. Stephani, *Differential Equations. Their Solution Using Symmetries*, Cambridge University Press 1989
- [25] L. Arnold, *Random Dynamical Systems*, Springer 1988
- [26] L.C. Evans, *An Introduction to Stochastic Differential Equations*, A.M.S. 2013
- [27] D. Freedman, *Brownian Motion and Diffusion*, Springer 1983
- [28] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North Holland 1981
- [29] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, North Holland 1992; 2003
- [30] B. Oksendal, *Stochastic Differential Equations* (4<sup>th</sup> edition), Springer 1985
- [31] D.W. Stroock, *Markov Processes from K.Ito’s Perspective*, Princeton UP 2003

## SUPPLEMENTARY MATERIAL

This file provides supplementary material for the paper ‘‘Symmetry classification of scalar Ito equations with multiplicative noise’’, referred afterwards as ‘‘the main paper’’. In particular we detail the computations needed to prove the second part of Theorem 1 in there; in the following when we refer to Theorem 1 we mean Theorem 1 in that paper.

Here we examine all the different possibilities (with some duplication; i.e. the same situation is met under different headings, see also Table A below) to have a solution to the determining equations

$$\varphi_t + f \varphi_x - \varphi f_x + \frac{1}{2} \Delta(\varphi) = 0 \quad (34)$$

$$\varphi_w + \sigma \varphi_x - \varphi \sigma_x = R \sigma, \quad (35)$$

where  $\Delta$  is the Ito Laplacian, which in our scalar case reads simply

$$\Delta(\Psi) := \frac{\partial^2 \Psi}{\partial w^2} + 2\sigma \frac{\partial^2 \Psi}{\partial x \partial w} + \sigma^2 \frac{\partial^2 \Psi}{\partial x^2}. \quad (36)$$

We identify all of these possibilities with one of the cases listed in Theorem 1.

We have already constrained the forms of  $f$  and  $\varphi$  to those appearing in equations (22) and (23) in the main paper (we will use this notation throughout). We now identify all the possible solutions of (34) and (35) by refining the time dependence of the free parameters in  $f$  and  $\varphi$ . The relevant point is that in the equations above all dependencies on  $x$  can now be made explicit and thus the coefficients of different monomials in  $\{z, x, \log x\}$  should vanish separately.

The computation is organized according to  $\alpha(t)$ ; we first consider the generic case, i.e. the case where  $\alpha(t) \neq \{-1, 0+1\}$ , and then consider also the different special cases  $\alpha(t) = \{-1, 0+1\}$ . Note that, by Remark 8 of the main paper, in the special cases we can always set  $\eta(t) = 0$ .

### A Generic case

We first require the coefficient of the term  $x^{1+\alpha} \log(x)$  to vanish; this means

$$-2 \exp[\alpha(t) S(t) z] \alpha(t) S(t) \eta(t) R = 0. \quad (37)$$

Recalling that  $S(t) \neq 0$  (or we would have a deterministic rather than a stochastic equation), and that  $\alpha(t) \neq 0$  (or we would be in one of the special cases), we are left with the options

- **(G.a)**  $R = 0, \eta(t) \neq 0$ ;
- **(G.b)**  $\eta(t) = 0, R \neq 0$ ;
- **(G.ab)**  $R = 0, \eta(t) = 0$ .

Note that the case  $R = 0$  yields only standard symmetries, while conversely W-symmetry will occur only in the case (G.b).

### A.1 Case (G.a): $R = 0, \eta \neq 0$ .

Let us first consider the case

$$R = 0, \quad \eta(t) \neq 0. \quad (38)$$

Then (37) is satisfied, and when we look at the coefficient of  $x^{1+\alpha(t)}$  we get

$$-2 \exp[\alpha(t) S(t) z] \alpha(t) S(t) \eta(t) Q_3(t). \quad (39)$$

For this to vanish – as neither one of  $S(t)$ ,  $\alpha(t)$  and  $\eta(t)$  can be zero under the present assumptions – we need that

$$Q_3(t) = 0, \quad (40)$$

and we assume this. The equation (37) is now a quadratic expression in  $x$  (also depending on  $z$ ), call it

$$\sum_{k=0}^2 E_k x^k = 0,$$

and this requires the  $E_k$  to vanish separately; by looking at coefficients of  $x^2$  and of  $x^0$ , we get

$$\begin{aligned} E_0 &= -2 \alpha(t) [1 - \alpha^2(t)] [1 + \alpha(t)] S(t) F_1(t) Q_2(t), \\ E_2 &= 2 \alpha(t) [1 - \alpha^2(t)] [1 - \alpha(t)] S(t) F_3(t) Q_2(t). \end{aligned}$$

Recalling that  $S(t) \neq 0$  and that we assumed  $\alpha(t) \neq \{-1, 0 + 1\}$ , we have two options, viz.

- **(G.a.1)**  $Q_2(t) = 0$ ;
- **(G.a.2)**  $F_1(t) = F_3(t) = 0$ .

In case **(G.a.1)**, the equation (34) is now fully satisfied; but this also enforces  $\varphi = 0$ , i.e. we get no symmetry altogether.

We will then consider case **(G.a.2)**, i.e. assume

$$F_1(t) = F_3(t) = 0; \quad Q_2(t) \neq 0. \quad (41)$$

Now the equation (34) is homogeneous of degree one in  $x$ , and linear in  $z$ .

The coefficient of the term  $xz$  reads

$$2 \alpha(t) [1 - \alpha^2(t)] S(t) Q_2(t) [S(t) \alpha'(t) + S'(t) \alpha(t)].$$

As  $Q_2(t) \neq 0$ , we require that

$$S(t) \alpha'(t) + S'(t) \alpha(t) = 0,$$

which yields immediately

$$\alpha(t) = \frac{k_1}{S(t)}. \quad (42)$$

We stress that we should exclude both  $k_1 = 0$  and the case where  $S(t)$  is constant,  $S(t) = s_0 \neq 0$  and  $s_0 = \pm k_1$ , as these would lead to  $\alpha(t) = \pm 1$ .

In this way (34) is reduced to

$$k_1 x (k_1^2 - S^2(t)) [k_1 S^3(t) Q_2(t) + 2 k_1 Q_2(t) S'(t) + 2 S(t) Q_2(t) (S'(t) - k_1 F_2(t)) - 2 S^2(t) Q_2'(t)] = 0 .$$

This can be solved for  $F_2(t)$ , yielding

$$F_2(t) = \frac{S^2(t)}{2} + \frac{S'(t)}{k_1} + \frac{S'(t)}{S(t)} - \frac{S(t)}{k_1} \frac{Q_2'(t)}{Q_2(t)} . \quad (43)$$

In this case we obtain (recall  $k_1 \neq 0$ )

$$f(x, t) = x \left[ S^2(t) \left( \frac{1}{2} + \frac{x^{k_1/S(t)} \eta(t)}{k_1 (k_1^2 - S^2(t))} \right) + \frac{S'(t)}{k_1} + \frac{\log(x) S'(t)}{S(t)} - \frac{S(t) Q_2'(t)}{k_1 Q_2(t)} \right] . \quad (44)$$

This expression gets simplified for  $S(t) = s_0 \neq \pm k_1$ ; in this case we get

$$f(x, t) = x \left[ s_0^2(t) \left( \frac{1}{2} + \frac{x^{k_1/s_0} \eta(t)}{k_1 (k_1^2 - s_0^2)} \right) - \frac{s_0 Q_2'(t)}{k_1 Q_2(t)} \right] . \quad (45)$$

As for the symmetries, in this case we get

$$q(t, z) = \exp[-k_1 z] Q_2(t) ; \quad (46)$$

going back to the original variables and the full symmetry coefficient, we get

$$\varphi(x, t, w) = x^{[1+k_1/S(t)]} e^{-k_1 w} Q_2(t) . \quad (47)$$

This shows that in this case we get nontrivial symmetries; these are (random) standard ones, as (they depend on  $w$  and)  $R = 0$ .

This case corresponds to case (a) in the list of Theorem 1.

## A.2 Case (G.b): $\eta = 0$ , $R \neq 0$ .

We will now look at case (G.b), i.e. assume

$$R \neq 0 , \quad \eta(t) = 0 . \quad (48)$$

Substituting for (48) in (34) we obtain an expression quadratic in  $x$  and also depending linearly on  $\log(x)$ ,  $z$  and on  $\exp[\alpha(t)S(t)z]$ . The coefficient of  $\exp[\alpha(t)S(t)z] \log(x)$  is

$$-2 R \alpha(t) [1 - \alpha^2(t)] S(t) [F_1(t) - x^2 F_3(t)] .$$

If we exclude the special cases  $\alpha(t) = \{-1, 0, +1\}$ , recall that  $S(t) \neq 0$ , and exclude also  $R = 0$ , for this to vanish identically in  $x$  we need again

$$F_1(t) = 0 = F_3(t) , \quad (49)$$

which hence we assume. This eliminates any dependence on  $\log(x)$

Next we look at the coefficient of  $z$  in (34), which is now

$$2x\alpha(t)[1 - \alpha^2(t)]S(t)Q_2(t) [S(t)\alpha'(t) + S'(t)\alpha(t)] .$$

Requiring this to vanish, we have two cases:

- **(G.b.1)**  $Q_2(t) = 0$ ;
- **(G.b.2)**  $\alpha(t) = k_1/S(t)$  where  $k_1 \neq 0$  is a constant.

These cases look identical to subcases (G.a.1) and (G.a.2) above, but now we mean that (G.b) is also satisfied, while in there it was assumed that (G.a) holds. Moreover, now we do not have (40), so  $Q_2$  can be zero without implying  $q(t, z) = 0$ .

### A.2.1 Case (G.b.1)

Case **(G.b.1)** is rather simple: the equation (34) is then reduced to

$$-\exp[z\alpha S]x\alpha(1 - \alpha^2) [RS^3 + 2RSF_2 - 2RS' - 2Q_3S' + 2SQ_3'] = 0 .$$

We can drop the initial factors and consider just the term in square brackets; moreover we write

$$Q_3(t) = R\theta(t) ; \quad (50)$$

hence

$$R [S^3(t) + 2(S(t)F_2(t) - S'(t)) - 2(\theta(t)S'(t) + S(t)\theta'(t))] = 0 . \quad (51)$$

This – and hence the full equation (34) – is solved by

$$F_2(t) = -\frac{S^3(t) - 2(1 + \theta'(t))S'(t) + 2S(t)\theta'(t)}{2S(t)} . \quad (52)$$

In this way we obtained

$$f(x, t) = -x \frac{S^3(t) - 2(\log(x) + \theta(t))S'(t) + 2S(t)\theta'(t)}{2S(t)} . \quad (53)$$

For  $S(t) = s_0$  this reduces to

$$f(x, t) = -\frac{x}{2} [s_0^2 + 2\theta'(t)] . \quad (54)$$

As for the symmetries, in this case we get

$$q(t, z) = R\theta(t) , \quad (55)$$

and hence in the original variables

$$\varphi(x, t, w) = Rx [\log(x) + \theta(t)] . \quad (56)$$

This corresponds to case **(e)** in the list of Theorem 1, with  $\mathcal{F}(t) = 0$ .

### A.2.2 Case (G.b.2)

Let us now consider case (G.b.2). Now the equation (34) is linear in  $x$  and also depends on  $\exp[k_1 z]$ . Looking at the coefficient of  $x \exp[k_1 z]$  and dropping a coefficient  $k_1(k_1^2 - S^2)/S^3$ , and using again (50), we are left with the equation

$$S' - 2(1 + \theta)S' + 2S(F_2 + \theta') = 0 ,$$

which is solved by

$$F_2(t) = \frac{-S^3(t) + 2S'(t) + 2\theta(t)S'(t) - 2S(t)\theta'(t)}{2S(t)} . \quad (57)$$

With this, (34) becomes homogeneous of degree one in  $x$ ; dropping an inessential factor  $2k_1 R x(k_1^2 - S^2(t))$ , we are left with

$$k_1 S^3 Q_2 - k_1 Q_2 \theta S' - S^2 Q_2' + S Q_2 (S' + k_1 \theta') = 0 .$$

This, and hence the full equation (34) – is solved by

$$\begin{aligned} Q_2(t) &= k_2 \exp \left[ \int_0^t \frac{k_1 S^3(y) + S(y)S'(y) - k_1 [\theta(y)S'(y) + \theta'(y)S(y)]}{S^2(y)} dy \right] \\ &:= H(t) . \end{aligned} \quad (58)$$

Note that for  $S(t) = s_0$  this reduces to

$$H(t) = H_0(t) := k_1 \left[ s_0 + \frac{\theta'(t)}{s_0} \right] . \quad (59)$$

In this way we obtain

$$f(x, t) = -x \frac{S^3(t) - 2 [\log(x) + \theta(t)] S'(t) + 2S(t)\theta'(t)}{2S(t)} ; \quad (60)$$

for  $S(t) = s_0$  this reduces to

$$f(x, t) = -x \left[ \frac{s_0^2}{2} + \theta'(t) \right] . \quad (61)$$

As for the symmetries, writing  $k_2 = Rk_3$ , we get

$$q(t, z) = R [k_3 \exp [H(t) - k_1 z] + \theta(t)] , \quad (62)$$

and hence

$$\varphi(x, t, w) = R x \left[ k_3 \exp [H(t) - k_1 w] x^{k_1/S(t)} + \log(x) + \theta(t) \right] . \quad (63)$$

This corresponds to case (e) in the list of Theorem 1.

### A.3 Case (G.ab): $R = 0 = \eta(t)$

In our discussion we have not allowed, so far, to have both  $R = 0$  and  $\eta(t) = 0$ ; this represents the intersection of the two cases considered above. When we set

$$R = 0, \quad \eta(t) = 0,$$

the equation (34) is of course simpler than in previous cases. Looking at the coefficient of  $x^2 \exp[-\alpha(t)S(t)z]$  we get

$$2 \alpha(t) [1 - \alpha^2(t)] S(t) F_3(t) Q_3(t) = 0.$$

We have again two cases:

- (G.ab.1)  $F_3(t) = 0$ ;
- (G.ab.2)  $Q_3(t) = 0$ .

#### A.3.1 Case (G.ab.1)

Let us first consider the case (G.ab.1). Now the coefficient of  $x \exp[-\alpha(t)S(t)z]$  vanishes for

$$Q_3(t) = k_3 S(t); \quad (64)$$

imposing this and looking at the coefficient of  $\exp[-\alpha(t)S(t)z]$  we obtain that moreover

$$F_1(t) = 0. \quad (65)$$

At this point the equation (34) is homogeneous of degree one in  $x$ , and linear in  $z$ , Dropping an overall factor  $x\alpha(t)[1 - \alpha^2(t)]$  and looking at terms of degree one in  $z$  we get

$$-2 S(t) Q_2(t) [S(t) \alpha'(t) + S'(t) \alpha(t)] = 0.$$

This is solved either for  $Q_2(t) = 0$  or for  $\alpha(t) = k_1/S(t)$ .

*Case (G.ab.1.1).* For  $Q_2(t) = 0$  the full equation is also solved. In this case we get

$$f(x, t) = x F_2(t) + x [\log(x) - 1] \frac{S'(t)}{S(t)}, \quad (66)$$

which for  $S(t) = s_0$  just gives

$$f(x, t) = x F_2(t). \quad (67)$$

As for symmetries, we get

$$q(t, z) = k_3 S(t), \quad (68)$$

and therefore

$$\varphi(x, t, w) = k_3 x S(t). \quad (69)$$

This corresponds to case (c) in Theorem 1.

*Case (G.ab.1.2).* In the case  $\alpha(t) = k_1/S(t)$ , the equation (34) is finally solved setting

$$F_2(t) = \frac{S^2(t)}{2} + \frac{S'(t)}{k_1} + \frac{S'(t)}{S(t)} - \frac{S(t) Q_2'(t)}{k_1 Q_2(t)} .$$

In this case we get

$$f(x, t) = \left[ S^2(t) + \frac{2}{k_1} \left( S'(t) - S(t) \frac{Q_2'(t)}{Q_2(t)} \right) + 2 \frac{S'(t)}{S(t)} \log(x) \right] \frac{x}{2} , \quad (70)$$

which for  $S(t) = s_0$  reduces to

$$f(x, t) = \left[ s_0^2 - \frac{2s_0}{k_1} \frac{Q_2'(t)}{Q_2(t)} \right] \frac{x}{2} . \quad (71)$$

As for symmetries, we get

$$q(t, z) = k_3 S(t) + e^{-k_1 z} Q_2(t) , \quad (72)$$

and hence

$$\varphi(x, t, w) = k_3 S(t) x + Q_2(t) e^{-k_1 w} x^{[1+k_1/S(t)]} . \quad (73)$$

This corresponds to case (b) in Theorem 1.

### A.3.2 Case (G.ab.2)

Let us now consider the case **(G.ab.2)**, i.e.  $Q_3(t) = 0$ . We have to annihilate an expression which is linear in  $z$ . The coefficient of  $z$  is just

$$2 x \alpha(t) [1 - \alpha^2(t)] S(t) Q_2(t) [S(t)\alpha'(t) + S'(t)\alpha(t)] .$$

If we set  $Q_2(t) = 0$  the full equation is satisfied, but we get  $\varphi(x, t, w) = 0$ , i.e. no symmetry.

By setting  $\alpha(t) = k_1/S(t)$  we are reduced to consider an expression quadratic in  $x$  and not depending on  $z$ . The coefficients of  $x^2$  and  $x^0$  vanish if we require either  $Q_2(t) = 0$  or  $F_1(t) = F_3(t) = 0$ . In the case  $Q_2(t) = 0$  the full equation is satisfied, but  $\varphi(x, t, w) = 0$ . So we set

$$F_1(t) = 0 = F_3(t) .$$

In this way we are left with an equation, homogeneous of degree one in  $x$ , which is solved by

$$F_2(t) = \frac{S^2(t)}{2} - \frac{S(t) Q_2'(t)}{k_1 Q_2(t)} + \left( \frac{1}{k_1} + \frac{1}{S(t)} \right) S'(t) . \quad (74)$$

This yields in turn

$$f(x, t) = \frac{1}{2} x \left[ S^2(t) - \frac{2 S(t) Q_2'(t)}{k_1 Q_2(t)} + \frac{2}{k_1} S'(t) + \frac{2 \log(x) S'(t)}{S(t)} \right], \quad (75)$$

which for  $S(t) = s_0$  reduces to

$$f(x, t) = \frac{1}{2} s_0 x \left[ s_0 - \frac{2}{k_1} \frac{Q_2'(t)}{Q_2(t)} \right]. \quad (76)$$

As for symmetries, we have

$$q(t, z) = \exp[-k_1 z] Q_2(t), \quad (77)$$

and hence

$$\varphi(x, t, w) = x^{[1+k_1/S(t)]} \exp[-k_1 w] Q_2(t). \quad (78)$$

This corresponds to case **(b)** in Theorem 1.

We have thus concluded the discussion for the general case. We will now pass to consider special cases.

## B Special case P: $\alpha(t) = +1$

For  $\alpha(t) = 1$ , the equation (34) contains a factor  $\exp[-S(t)z]x^2 \log^2(x)$ ; the vanishing of its coefficient corresponds to

$$-2 R \eta(t) = 0,$$

hence we must have one of the cases **(a)**, **(b)**, **(ab)** considered in the generic setting. These will now be dubbed as cases (P.a), (P.b) and (P.ab) respectively.

### B.1 Case (P.a)

It follows from Remark 8 of the main paper that we can disregard the case  $\eta(t) \neq 0$ ; we will however perform computations in this case as well, to show we obtain no possibility of a symmetric equation.

Let us first set

$$R = 0, \quad \eta(t) \neq 0.$$

The coefficient of  $\exp[-S(t)z]x^2 \log(x)$  is

$$-2 \eta(t) Q_3(t). \quad (79)$$

Thus we require

$$Q_3(t) = 0,$$

and we have an equation linear in  $z$ . The vanishing of the coefficient of  $xz$  in this requires

$$4 S(t) Q_2(t) S'(t) = 0.$$

Choosing  $Q_2(t) = 0$  satisfies the full equation but produces  $\varphi = 0$ . Choosing instead

$$S(t) = s_0 \neq 0$$

gives an equation quadratic in  $x$ . For this to hold – not allowing  $Q_2(t) = 0$ , which would give no nontrivial symmetry – we would have to set

$$\eta(t) = 0, \quad F_1(t) = 0, \quad F_2(t) = \frac{s_0^2}{2} - \frac{Q_2'(t)}{Q_2(t)}.$$

But this is in contradiction with our assumption  $\eta(t) \neq 0$ . Thus in this case **(P.a)** we have no nontrivial symmetries.

## B.2 Case (P.b)

Let us consider case **(P.b)**, i.e. the case where

$$\eta(t) = 0, \quad R \neq 0. \quad (80)$$

Looking at coefficients of terms involving  $\exp[zS(t)] \log(x)$ , we see that we need

$$F_1(t) = 0 = F_3(t).$$

With this the equation is homogeneous of degree one in  $x$ , so we drop the inessential overall  $2x$  factor. The coefficient of  $z$  in the new equation is

$$- 2 S(t) Q_2(t) S'(t),$$

so we have either  $Q_2(t) = 0$ , which we call case **(P.b1)**, or  $S(t) = s_0$ , which we call case **(P.b2)**.

*Case (P.b1).* For

$$Q_2(t) = 0$$

it turns out we also have to require

$$F_2(t) = - \frac{R S^3(t) - 2[R + Q_3(t)] S'(t) + 2 S(t) Q_3'(t)}{2 R S(t)}. \quad (81)$$

In this way we obtain (once again  $Q_3 = R\theta$ )

$$f(x, t) = - \left( \frac{S^2(t)}{2} - [\log(x) + \theta(t)] \frac{S'(t)}{S(t)} + \theta'(t) \right) x, \quad (82)$$

$$\varphi(x, t, w) = R [\log(x) + \theta(t)] x. \quad (83)$$

This corresponds to case **(e)** of Theorem 1.

*Case (P.b2).* If instead we set

$$S(t) = s_0 \neq 0,$$

then we also need (we write again  $Q_3 = R\theta$ )

$$F_2(t) = - \frac{s_0^2 + \theta'(t)}{2}$$

and moreover

$$Q_2(t) = k_1 \exp [s_0^2 t + \theta(t)] .$$

In this way we get (writing also  $k_1 = Rk_3$ )

$$f(x, t) = - \left( \frac{s_0^2}{2} + \theta'(t) \right) x \quad (84)$$

$$\varphi(x, t, w) = R (k_3 \exp [s_0^2 t - s_0 w + \theta(t)] x^2 + x \log(x) + \theta(t) x) \quad (85)$$

This corresponds to case (h) of Theorem 1.

### B.3 Case (P.ab)

If we set

$$R = 0 , \quad \eta(t) = 0 ,$$

we obtain an equation which does not depend on  $\log(x)$ , but still has  $\exp[S(t)z]$  factors. In particular, looking at coefficients of  $\exp[S(t)z]x^2$  and  $\exp[S(t)z]$  we must require

$$\begin{aligned} 4 S(t) F_1(t) Q_3(t) &= 0 , \\ 4 S(t) F_3(t) Q_3(t) &= 0 ; \end{aligned}$$

thus we have the two cases

- (P.ab.1)  $F_1(t) = F_3(t) = 0$ ,
- (P.ab.2)  $Q_3 = 0$ .

#### B.3.1 Case (P.ab.1)

Setting

$$F_1(t) = 0 = F_3(t)$$

and looking at the coefficient of  $\exp[S(t)z]x$  we obtain

$$4 [S(t) Q_3'(t) - S'(t) Q_3(t)] = 0 ,$$

which requires

$$Q_3(t) = k_1 S(t) ;$$

note that here  $k_1 = 0$  is a legitimate choice.

With this, (34) is homogeneous of degree one in  $x$  and linear in  $z$ . The coefficient of  $xz$  is

$$-4 S(t) Q_2(t) S'(t) ,$$

hence either  $Q_2(t) = 0$  or  $S(t) = s_0$  is a constant.

*Case (P.ab.1.1).* For

$$Q_2(t) = 0$$

the full equation is solved; now  $F_2(t)$  is unconstrained, and we get

$$f(x, t) = \left[ F_2(t) - \frac{S'(t)}{S(t)} \right] x + \left( \frac{S'(t)}{S(t)} \right) x \log x, \quad (86)$$

$$\varphi(x, t, w) = k_1 S(t) x. \quad (87)$$

This corresponds to case (c) of Theorem 1.

*Case (P.ab.1.2).* For  $S(t) = s_0$  the equation is reduced to

$$2x [s_0^3 Q_2(t) - 2s_0 (F_2(t) Q_2(t) + Q_2'(t))] = 0,$$

which is solved by

$$F_2(t) = \frac{s_0^2}{2} - \frac{Q_2'(t)}{Q_2(t)}.$$

In this way we obtain

$$f(x, t) = \left( \frac{s_0^2}{2} - \frac{Q_2'(t)}{Q_2(t)} \right) x, \quad (88)$$

$$\varphi(x, t, z) = e^{-s_0 w} Q_2(t) x^2 + k_1 s_0 x. \quad (89)$$

This corresponds to case (f) of Theorem 1.

### B.3.2 Case (P.ab.2)

If after choosing  $R = 0$  and  $\eta(t) = 0$  we do not set  $F_1 = F_3 = 0$  but instead

$$Q_3(t) = 0,$$

we are left with an equation linear in  $z$ . The coefficient of  $xz$  is just

$$-4 S(t) Q_2(t) S'(t).$$

so either  $S(t)$  is a constant or

$$Q_2(t) = 0.$$

Choosing this the equation is satisfied, but we get  $\varphi(x, t, w) = 0$ , hence no symmetry.

Choosing instead

$$S(t) = s_0 \neq 0,$$

we are left with the two equations

$$\begin{aligned} 8 s_0 F_1(t) Q_2(t) &= 0 , \\ 4 s_0 Q_2'(t) - 2 [s_0^3 - 2s_0 F_2(t)] Q_2(t) &= 0 . \end{aligned}$$

The case  $Q_2(t) = 0$  was just considered, so we exclude this and consider instead the other option, i.e.

$$F_1(t) = 0 , \quad F_2(t) = \frac{s_0^2}{2} - \frac{Q_2'(t)}{Q_2(t)} . \quad (90)$$

In this way we obtain

$$\begin{aligned} f(x, t) &= \left( \frac{s_0^2}{2} - \frac{Q_2'(t)}{Q_2(t)} \right) x + F_3(t) x^2 , \\ \varphi(x, t, w) &= Q_2(t) \exp[-s_0 w] x^2 . \end{aligned} \quad (91) \quad (92)$$

This corresponds to case **(f)** of Theorem 1.

This concludes the discussion for the special case P.

**Remark.** In a recent work (see ref.[1] of the main paper) the symmetries of the stochastic logistic equation

$$dx = (\alpha x - \beta x^2) dt + \gamma dw$$

(with  $\alpha, \beta, \gamma$  positive real constants) have been determined and used to explicitly integrate it. In particular, it turned out that in this case

$$\varphi = \exp [ - ((\alpha - \gamma^2/2)t + \gamma w) ] x^2 .$$

This fits within the case last considered, (P.ab.2), with the choice  $Q_2(t) = 1$ ,  $F_3(t) = -\beta$ ; note that here  $s_0 = \gamma$ .  $\odot$

## C Special case M: $\alpha(t) = -1$

When we set

$$\alpha(t) = -1 , \quad (93)$$

and look for the vanishing of the coefficient of  $\log^2(x)$  in (34), we get

$$R \eta(t) = 0 . \quad (94)$$

We have of course two cases: **(M.a)**  $R = 0$ , and **(M.b)**  $\eta(t) = 0$ ; we will consider the intersection of these two as a separate case **(M.ab)** for ease of discussion.

**C.1 Case (M.a):**  $R = 0, \eta(t) \neq 0$ .

Looking at the coefficient of  $\log(x)$ , we have

$$\eta(t) Q_3(t) = 0 ,$$

as we do not allow at present  $\eta(t) = 0$ , we set

$$Q_3(t) = 0 .$$

The whole equation has now a factor  $\exp[zS(t)]$ , which we drop from now on.

We are left with an expression quadratic in  $x$ ; the term independent of  $x$  is just

$$- \eta(t) Q_2(t) ,$$

hence we set

$$Q_2(t) = 0 .$$

The equation is now satisfied but  $\varphi = 0$ , so we have no symmetry.

**C.2 Case (M.b):**  $\eta(t) = 0, R \neq 0$ .

We now consider the case with  $R = 0$  and  $\eta(t) \neq 0$ . The coefficients of terms with  $\log(x)$  enforce

$$F_1(t) = 0 = F_3(t) .$$

In this way we have an expression homogeneous of degree one in  $x$  (we drop from now on the  $x$  factor), linear in  $z$  and also depending on  $\exp[zS(t)]$ .

The coefficient of  $z \exp[zS(t)]$  is just

$$2 S(t) Q_2(t) S'(t) ,$$

so either  $Q_2(t) = 0$  (case (M.b.1)) or  $S(t) = s_0 \neq 0$  (case (M.b.2)).

*Case (M.b.1).* Setting

$$Q_2(t) = 0$$

we are left with an equation which is solved by

$$F_2(t) = \frac{S'(t)}{S(t)} - \frac{1}{2} S(t) + \frac{Q_3(t)}{R} \frac{S'(t)}{S(t)} - \frac{Q_3'(t)}{R} .$$

We thus obtain (writing as usual  $Q_3 = R\theta$ )

$$f(x, t) = x \log(x) \frac{S'(t)}{S(t)} + x \left[ \theta(t) \frac{S'(t)}{S(t)} - \theta'(t) - \frac{S^2(t)}{2} \right] \quad (95)$$

$$\varphi(x, t, w) = R (\log(x) + \theta(t)) x . \quad (96)$$

This corresponds to case (e) of Theorem 1.

Case (M.b.2). If we set instead

$$S(t) = s_0 \neq 0 ,$$

the coefficient of the term  $\exp[s_0 z]$  vanishes for

$$F_2(t) = \frac{s_0^2}{2} + \frac{Q_2'(t)}{Q_2(t)} ;$$

the full equation is then solved (using again  $Q_3 = R\theta$ ) by

$$Q_2(t) = k_1 \exp[-s_0^2 t - \theta(t)] .$$

In this case we obtain

$$\begin{aligned} f(x, t) &= \left[ \frac{s_0^2}{2} - (s_0^2 + \theta'(t)) \right] x & (97) \\ \varphi(x, t, w) &= R (k_3 \exp[-s_0^2 t + s_0 w - \theta(t)] + x \log(x) + \theta(t) x) & (98) \end{aligned}$$

This corresponds to case (h) of Theorem 1.

### C.3 Case (M.ab): $R = 0$ , $\eta(t) = 0$ .

Finally we consider the subcase with

$$R = 0 , \quad \eta(t) = 0 .$$

The coefficient of  $\exp[zS(t)]x^2$  is

$$- 4 S(t) F_3(t) Q_2(t) .$$

Case (M.ab.1). Setting

$$Q_2(t) = 0$$

and looking at the coefficient of  $x$  we get

$$Q_3(t) = k_1 S(t) .$$

For  $k_1 = 0$  the equation is solved but  $\varphi = 0$ ; so we assume  $k_1 \neq 0$ . Then we also have to impose

$$F_1(t) = F_3(t) = 0 .$$

In this way we get

$$f(x, t) = \left( F_2(t) + \frac{S'(t)}{S(t)} \right) x + \left( \frac{S'(t)}{S(t)} \right) x \log(x) , \quad (99)$$

$$\varphi(x, t, w) = k_1 S(t) x . \quad (100)$$

This corresponds to case (c) of Theorem 1.

*Case (M.ab.2).* Let us now consider instead the choice  $F_3(t) = 0$  (rather than  $Q_2(t) = 0$  as above). Then the coefficient of  $\exp[zS(t)]xz$  yields

$$2 S(t) Q_2(t) S'(t) = 0 ,$$

so we have the two options  $Q_2(t) = 0$  or  $S(t) = s_0 \neq 0$ .

*Case (M.ab.2.1).* Choosing  $Q_2(t) = 0$  we are left with the two equations

$$S(t) F_1(t) Q_3(t) = 0 , \quad S(t) Q_3'(t) - S'(t) Q_3(t) = 0 .$$

The case  $Q_3(t) = 0$  is of no interest here, (it yields  $\varphi = 0$ ) so we set

$$F_1(t) = 0 , \quad Q_3(t) = k_1 S(t) .$$

In this way we obtain

$$f(x, t) = \left( F_2(t) + \frac{S'(t)}{S(t)} \right) x + \frac{S'(t)}{S(t)} x \log(x) , \quad (101)$$

$$\varphi(x, t, w) = k_1 S(t) x . \quad (102)$$

This corresponds to case (c) of Theorem 1.

*Case (M.ab.2.2).* Finally, choosing  $S(t) = s_0 \neq 0$  (instead of  $Q_2(t) = 0$ ) we get

$$F_2(t) = \frac{s_0^2}{2} + \frac{Q_2'(t)}{Q_2(t)} ;$$

this in turn enforces

$$F_1(t) Q_3(t) = 0 , \quad Q_3'(t) = 0 .$$

For  $Q_3(t) = 0$  we get

$$f(x, t) = F_1(t) + \left( \frac{s_0^2}{2} + \frac{Q_2'(t)}{Q_2(t)} \right) x , \quad (103)$$

$$\varphi(x, t, w) = \exp[s_0 w] Q_2(t) . \quad (104)$$

This corresponds to case (g) of Theorem 1.

For  $F_1(t) = 0$ ,  $Q_3(t) = k_1$  we get instead

$$f(x, t) = \left( \frac{s_0^2}{2} + \frac{Q_2'(t)}{Q_2(t)} \right) x , \quad (105)$$

$$\varphi(x, t, w) = k_1 x + \exp[s_0 w] Q_2(t) . \quad (106)$$

This also corresponds to case (g) of Theorem 1.

This concludes the discussion for the special case M.

## D Special case O: $\alpha(t) = 0$

The last special case we have to consider is that with

$$\alpha(t) = 0 .$$

Setting this, and looking at the coefficients of terms with  $\log(x)$ , we get that either  $R = 0$  or  $F_1(t) = F_3(t) = 0$ ; we denote these as cases (O.a) and (O.b) respectively. We will also consider case (O.ab) with  $R = F_1(t) = F_3(t) = 0$ .

### D.1 Case (O.a)

Setting  $R = 0$  and looking at terms with  $x^2$  and with  $x^0$ , we get

$$\begin{aligned} 2 S(t) F_3(t) [Q_2(t) + Q_3(t)] &= 0 \\ 2 S(t) F_1(t) [Q_2(t) + Q_3(t)] &= 0 \end{aligned}$$

If we set  $F_1(t) = F_3(t) = 0$  we are actually in case (O.ab); so we disregard this option (for the time being) and set instead

$$Q_3(t) = -Q_2(t) .$$

This fully satisfies equation (34); but it also yields  $\varphi = 0$ , so in case (O.a) we have no symmetry.

### D.2 Case (O.b)

In case (O.b), in order to satisfy the equation it suffices to choose  $F_2(t)$  appropriately; this also fixes  $f(x, t)$ . Writing now

$$Q(t) = Q_2(t) + Q_3(t) = R \theta(t) ,$$

we get

$$\begin{aligned} f(x, t) &= -\frac{x}{2S(t)} [S^3(t) - 2S'(t)(\log(x) + \theta(t)) + 2S(t)\theta'(t)] \quad (107) \\ \varphi(x, t, w) &= R (\log(x) + \theta(t)) x . \quad (108) \end{aligned}$$

This corresponds to case (e) of Theorem 1.

### D.3 Case (O.ab)

Finally we consider the case with

$$R = 0 , \quad F_1(t) = F_3(t) = 0 .$$

This is solved by choosing

$$\eta(t) = \frac{\theta(t) S'(t) - \theta'(t) S(t)}{\theta(t)}$$

and in this way we get

$$f(x, t) = \left( F_2(t) + [\log(x) - 1] \frac{\theta'(t)}{\theta(t)} \right) x \quad (109)$$

$$\varphi(x, t, w) = \theta(t) x . \quad (110)$$

This corresponds to case (d) of Theorem 1.

This concludes the discussion for the special case O, and hence all of our discussion.

## E Conclusion

We have exhausted the full classification of equations in the form of eq.(3) in the main paper admitting random or simple W-symmetries, i.e. equations described by vector fields of the form (4) in the main paper. This was obtained by solving the determining equations (34), (35).

The result of our detailed discussion is summarized in the Tables provided in Theorem 1. It is maybe worth, for the ease of the reader, to summarize the correspondence between the cases (a) – (h) considered in those Tables and in the statement of the Theorem 1, and the cases considered for the sake of our classification in this Supplementary Material. This is done in the following Table A.

cases in Theorem 1	cases in this supplementary material
(a)	(G.a.2)
(b)	(G.ab.1.2), (G.ab.2)
(c)	(G.ab.1.1), (P.ab.1.1), (M.ab.1), (M.ab.2.1)
(d)	(O.ab)
(e)	(G.b.1), (G.b.2), (P.b1), (M.b.1), (O.b)
(f)	(P.ab.1.2), (P.ab.2)
(g)	(M.ab.2.2)
(h)	(P.b2), M.b.2)

**Table A.** Correspondence between cases of Theorem 1 of the main text and cases considered in the present Supplementary Material.