On the value of acquired information in gambling, thermodynamics and population dynamics

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Information and entropy

John von Neumann to Claude Shannon:

> *You should call it entropy for two reasons: first because that is what the formula is in statistical mechanics but second and more important, as nobody knows what entropy is, whenever you use the term you will always be at an advantage!*

Interpretation of the information rate

Kelly, 1956:

> *a gambler can use the knowledge given him by the received symbols to cause his money to grow exponentially. The maximum exponential rate of growth of the gambler's capital is equal to the rate of transmission of information over the channel.*

Economy, information and evolution

The growth of capital has parallel in the growth of populations
The currency of evolution is fitness, i.e., number of offspring
What is the connection with the information rate?

Economy, information and evolution

Genet. Res., Camb. (1961), **2**, *pp.* 127–140
With 2 text-figures
Printed in Great Britain

Natural selection as the process of accumulating genetic information in adaptive evolution*

BY MOTOO KIMURA
National Institute of Genetics, Mishima, Japan

(*Received* 3 *October* 1960)

Economy, information and evolution

It was demonstrated that the rate of accumulation of genetic information in adaptive evolution is directly proportional to the substitutional load, i.e. the decrease of Darwinian fitness brought about by substituting for one gene its allelic form which is more fitted to a new environment.

Analogies

Gambling	Thermodynamics	Populations
Currency unit	—	Individual
Gambler	Demon	—
Option	State	Type
Log Capital	Extracted Work	Log Population Size
Side Information	Measurement	Acquired information
Memory	Non-equilibrium	Inherited information

After Rivoire, 2015

A model of an evolving population

Donaldson-Matasci et al., 2010

- A population of N_t individuals, discrete generations, in a varying environment
- Environment X_t, phenotype Φ_t, $\boldsymbol{x} = (x_0, x_1, \ldots, x_t, \ldots)$
- Fitness: $\mathcal{F}(\phi, x)$: expected # of offspring with pheno ϕ in environment x
- Bet hedging: $b_t(\phi)$: probability to assign pheno ϕ to an offspring at generation t

Growth rate of population size:

$$\Lambda(b) = \frac{1}{\mathcal{T}} \left\langle \log \frac{\mathcal{N}_{\mathcal{T}}}{\mathcal{N}_0} \right\rangle = \frac{1}{\mathcal{T}} \sum_{\boldsymbol{x}} p_{\boldsymbol{x}} \sum_{t=0}^{\mathcal{T}-1} \log \left(\mathcal{F}(\phi, x_t) b_t(\phi) \right)$$

Kelly case

- No inheritance: $b_t(\phi)$ does not depend on $\phi_{t'}$ for $t' < t$
- Perfect selectivity: $\mathcal{F}(\phi, x) = K(x)\delta_{\phi,x}$ (can be relaxed: HACCOU AND IWASA, 1995)

Then

$$\begin{aligned}\Lambda(b) &= \sum_x p_x \log\left(K(x)b(x)\right) \\ &= \underbrace{\sum_x p(x)\log K(x)}_{\langle \log K\rangle} + \underbrace{\sum_x p(x)\log p(x)}_{-H(p)} - \underbrace{\sum_x p(x)\log\frac{p(x)}{b(x)}}_{D_{\mathrm{KL}}(p\|b)}\end{aligned}$$

Optimal strategy:

$$b^*(x) = p_x$$

Optimal growth rate:

$$\Lambda_{\mathrm{opt}} = \langle \log K\rangle - H(X)$$

“Fair” gambling: $K(x) = 1/p(x)$, optimal growth rate 0

Cues

- Assume there is a partially informative cue Y on the environment
- Joint probability $p(x, y) = p(x|y)p(y)$ for environment x and cue y
- Conditional probability $\pi(\phi|y)$ for pheno ϕ with cue y
- Growth rate:

$$
\begin{aligned}
\Lambda &= \sum_{x,y} p(x,y) \log \sum_{\phi} \pi(\phi|y)\mathcal{F}(\phi,x) \\
&= \sum_{x,y} p(x,y) \log \left[\pi(x|y)K(x)\right] \qquad \text{(Kelly)} \\
&= \sum_{x} p(x) \log K(x) + \underbrace{\sum_{x,y} p(x,y) \log p(x|y)}_{-H(X)+I(X;Y)} - \underbrace{\sum_{x,y} p(x,y) \log \frac{p(x|y)}{\pi(x|y)}}_{D_{\mathrm{KL}}(p(x|y)\|\pi(x|y))}
\end{aligned}
$$

Fitness value of cues

- Optimal growth rate with $\pi(x|y) = p(x|y)$:

$$\Delta\Lambda_{\text{opt}} = \Lambda_{\text{opt}}(X|Y) - \Lambda_{\text{opt}} = I(X;Y)$$

- More generally: optimal conditional strategy $\pi^*(x|y)$ and optimal unconditional strategy $\pi^*(x)$

$$\Delta\Lambda_{\text{opt}} = I(X;Y) - \Bigg[\underbrace{D_{\text{KL}}(p(x|y)\|\pi^*(x|y))}_{\text{with cues}} - \underbrace{D_{\text{KL}}(p(x)\|\pi^*(x))}_{\text{without cues}}\Bigg]$$

But it can be shown that

$$D_{\text{KL}}(p(x|y)\|\pi^*(x|y)) - D_{\text{KL}}(p(x)\|\pi^*(x)) \geq 0$$

- Thus the fitness value of cues is given by $I(X;Y)$

Analogy with work extraction

Vinkler et al., 2014, Rivoire, 2015

- A two-state system: $x \in \{0, 1\}$
- Energy: E_x, $E_0 = 0$, $E_1 = \epsilon_0$
- Equilibrium distribution: $p_x^{\text{eq}} = \text{e}^{-(E_x - F)/k_{\text{B}}T}$
- A "demon" can switch the states, gleaning ($\mathcal{W} > 0$) or providing ($\mathcal{W} < 0$) energy ΔE:

$$\mathcal{W} = -\Delta E_x = E_x - E_{1-x}$$

- In the absence of cues one expect to provide energy on average:

$$\langle \mathcal{W} \rangle_{\text{eq}} = \langle -\Delta E \rangle_{\text{eq}} = (p_1^{\text{eq}} - p_0^{\text{eq}})\, \epsilon_0 < 0$$

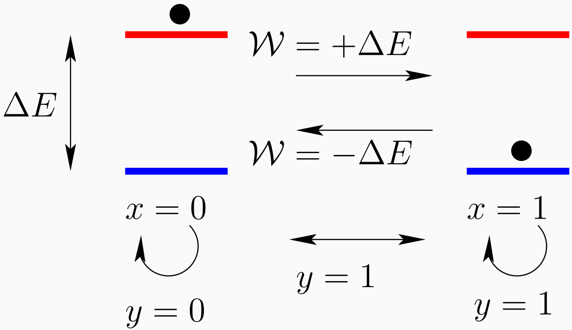
Analogy with work extraction

- In the presence of cues (measurement): $p(x|y)$ probability that the system is in x given measurement yields y, assume $p(x{=}y|y) > \frac{1}{2}$
- Switch ϕ: $x \xrightarrow{\phi=0} x$, $x \xrightarrow{\phi=1} (1-x)$

$$\Delta E_x(\phi) = E_1(x|\phi) - E_0(x) = (2x-1)\phi\,\epsilon_0$$

- Optimal conditional strategy: switch ($\phi = 1$) if $p(1|y) > \frac{1}{2}$, i.e., $\pi(\phi|y) = \delta_{\phi,y}$
- $\langle \mathcal{W} \rangle_{\text{opt}} = -\sum_{x,y} p(x|y)p(y)\Delta E_x(y)$

Analogy with work extraction



Analogy with work extraction

- Define $\pi(x|y) = \mathrm{e}^{-E_1(x|y)/k_\mathrm{B}T} / \underbrace{\sum_{x'} \mathrm{e}^{-E_1(x'|y)/k_\mathrm{B}T}}_{Z_1(y)}$
- Average extracted work:

$$\begin{aligned} \langle \mathcal{W} \rangle_{p(x,y)} &= -\langle \Delta E_x(y) \rangle_{p(x,y)} \\ &= k_\mathrm{B}T \sum_{x,y} p(x,y) \log \frac{\pi(x|y)}{p(x)} \qquad \text{(Kelly)} \\ &= k_\mathrm{B}T I(X;Y) - k_\mathrm{B}T \sum_y p(y) D_\mathrm{KL}(p(x|y) \| \pi(x|y)) \end{aligned}$$

- Can be generalized to more states: defining

$$F_1(y) = -k_\mathrm{B}T \log Z_1(y) \qquad F_0 = -k_\mathrm{B}T \log Z_0$$
$$\Delta F = \langle F_1(y) \rangle - F_0$$

one has

$$\langle \mathcal{W} \rangle \leq k_\mathrm{B}T I(X;Y) - \Delta F$$

Fluctuation relations

Cf. Hirono and Idaka, 2015 (gambling), Kobayashi and Sughiyama, 2015 (populations)

- From the analogy one can obtain fluctuation relations for the fluctuating growth rate:

$$\Lambda_t = \log \frac{\mathcal{N}_{t+1}}{\mathcal{N}_t}$$

- With a fixed strategy $b(x)$:

$$\mathrm{e}^{\Lambda_t} = K(x_t)b(x_t)$$

- Define $q(x)$ by $K(x) = K_0/q(x)$, $\sum_x q(x) = 1$, $s_X(x) = -p(x)$, $s_Q(x) = -q(x)$, then

$$\left\langle \mathrm{e}^{\Lambda + s - s_Q} \right\rangle = K_0 \sum_x b(x) = K_0$$

Fluctuation relations

- Note that for a Kelly optimal strategy b^* one has

$$\Lambda_t = K(x_t)p(x_t) = K_0 \frac{p(x_t)}{q(x_t)} = K_0 \mathrm{e}^{s_Q(x_t)-s(x_t)}$$

thus the exponent is given by the growth-rate loss:

$$\Lambda(b) - \log K_0 + s - s_Q = \Lambda(b) - \Lambda(b^*)$$

- In the presence of cues, with conditional strategy $\pi(x|y)$: defining $i_{x,y} = \log[p(x,y)/(p(x)p(y))]$

$$\left\langle \mathrm{e}^{\Lambda - \log K_0 + s_X - s_Q - i_{x,y}} \right\rangle = 1$$

which can be again expressed in terms of the growth-rate loss

Common and individual information

Rivoire and Leibler, 2011

- In thermodynamics and gambling, decisions are centralized
- In population dynamics, each individual has its own sensor
- $C(\psi|y)$ conditional probability of sensor output ψ with cue y
- $\pi_0(\phi|\psi)$ conditional probability of pheno ϕ with output ψ
- Optimal growth rate:

$$\Lambda_{\text{opt}} = \langle \log K \rangle + \sum_x p(x) \log \pi(x|x)$$

$$\pi(x|x) = \text{Prob}(\phi{=}x|x) = \sum_\psi \pi_0(x|\psi) C(\psi|x)$$

$$\sum_x p(x) \log \pi(x|x) = \underbrace{\sum_x p(x) \log \langle \pi_0(x|\psi) \rangle_{\Psi|X}}_{\text{annealed}} \geq \underbrace{\sum_{x,\psi} p(x,\psi) \log \pi_0(x|\psi)}_{\text{quenched}}$$

- Thus Λ_{opt} is larger when individual sensors are available

Inheritance:

- We assume that the environment is described by a Markov chain: $x \xrightarrow{W_{x'x}} x'$
- We focus on one time step: $X_0 \longrightarrow X_1$ (can be generalized to history dependence)
- The pdf of X_1 depends on both X_0 and the current cue Y_1: $P_X(X_1|X_0, Y_1)$
- The pheno $\Phi_1 = X_1$ also depends both on X_0 and on the cue Y_1: $\pi(X_1|X_0, Y_1)$
- The growth rate is given by

$$\Lambda = \langle \log K \rangle - H(X_1|X_0) + I(X_1; Y_1|X_0) \\ - \langle D_{\mathrm{KL}}(P(X_1|X_0, Y_1) \| \pi(X_1|X_0, Y_1)) \rangle_{P(X_0, X_1)}$$

Inheritance, memory and non-equilibrium

- Value of acquired information: optimal strategy $\pi^* = P(X_1|X_0, Y_1)$, optimal growth rate

$$\Lambda_{\text{opt}} = \langle \log K \rangle - H(X_1|X_0) + \underbrace{I(X_1; Y_1|X_0)}_{I_{\text{acquired}}}$$

- Value of inherited information: rewrite the growth rate

$$\Lambda = \langle \log K \rangle - H(X_1|Y_1) + I(X_1; X_0|Y_1) \\ - \langle D_{\text{KL}}(P(X_1|X_0, Y_1) \| \pi(X_1|X_0, Y_1)) \rangle_{P(X_0, X_1)}$$

- Optimal growth rate

$$\Lambda_{\text{opt}} = \langle \log K \rangle - H(X_1|Y_1) + \underbrace{I(X_1; X_0|Y_1)}_{I_{\text{inherited}}}$$

- Total information

$$I_{\text{tot}} = I(X_1; X_0) + I(X_1; Y_1|X_0) = I(X_1; Y_1) + I(X_1; X_0|Y_1)$$

N.B.: Inequalities hold in general for imperfect selectivity ($I_{\text{acquired}} \leq I(X_1; Y_1|X_0)$) etc.

Inheritance, memory and non-equilibrium

- Fluctuation theorem in population with inheritance:

$$\left\langle \mathrm{e}^{\Lambda - \log K_0 - s_Q - i_{x_1 x_0 y_1} + s_X - i_{x_0 y_1}} \right\rangle = 1$$

  Generalized by Kobayashi and Sughiyama by the consideration of the reverse trajectories...

- Interpretation of inheritance in stochastic thermodynamics:
  - System described by coordinate x, potential $E(x, \lambda)$
  - Noisy measurement of x made at regular intervals τ
  - A demon can act on λ to change $E(x)$ and extract work
  - τ smaller than equilibration time (Cf. Pal and Reuveni, 2017)
- One can consider optimizing the output power $\langle W \rangle / \tau$

Summary

- There is a deep analogy between the role of information acquisition in population dynamics, in gambling and in stochastic thermodynamics
- Some aspects of population dynamics are however peculiar to it, e.g., the role of distributed information (sensing)
- These analogies may help in identifying new relations in each of the different contexts
- I did not discuss *learning*, i.e., how the optimal strategies can be approached
    - For the case of gambling, this leads to portfolio management (Cover and Ordentlich, 1996)
    - For the case of evolution, to genotype selection with genotypes as bet-hedging strategies, subject to Darwinian selection (see, e.g., Xue and Leibler, 2016)
    - For the case of work extraction, to adaptive control protocol (Vinkler et al., 2015)

Thanks!

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