Erratum for “Random Lie-point symmetries of stochastic differential equations” (*J. Math. Phys.* 58 (2017), 053503)

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In our recent paper [1], due to a regrettable and rather trivial mistake, a term is missing in the expression (6) for the Ito Laplacian. The correct formula is, of course

$$\Delta u \ := \ \sum_{k=1}^n \frac{\partial^2 u}{\partial w^k \partial w^k} \ + \ \sum_{j,k=1}^n \left(\sigma \, \sigma^T\right) \ \frac{\partial^2 u}{\partial x^j \, \partial x^k} \ + \ 2 \ \sum_{j,k=1}^n \sigma^{ik} \ \frac{\partial^2 u}{\partial x^j \, \partial w^k} \ .$$

(The reader is alerted that the same mistake found its way into the recent review paper by one of the authors [2].)

This error has no consequence on our general discussion – conducted in terms of the Δ operator – except for Section VIII (see below); but it does affect the specific computations occurring in concrete examples and some side remarks.

In particular, the following simple amendments should be inserted in the paper as a consequence to the error in eq.(6):

1. The final part of **Remark 2** should just read “does now also include derivatives w.r.t. the w^k variables, which are of course absent in (9).”
2. In **Example 1**, the last five lines should read as follows: “Plugging this into the first equation, we get $F_t = 0$, hence $F = F(z)$ and any smooth function $\varphi(z)$ of $z = x - \sigma_0 t$ provides a simple random symmetry for (34). It should also be noted that $dz = 0$ on solutions to our equation (34), see Remark 6.”
3. In **Example 2**, the line after “we get two equations” should read as:

$$\psi + z\psi_z = 0 \ , \quad 2\psi_t - z\psi_z = 0 \ .$$

   (The conclusions, i.e. the lines below these equations, are correct.)

Note that Examples 3 & 4 are unaffected by the error in (6); in particular, concerning Example 3, any function $\eta(z_1, z_2, t)$ satisfies $\Delta(\eta) = 0$.

Moreover:

- A misprint was present in the last displayed equation of Example 3; this should read as follows:

$$\frac{\partial \eta_2}{\partial t} \ + \ a_2 \, \frac{\partial \eta_2}{\partial z_2} \ + \ \frac{a_1}{x_1} \, \frac{\partial \eta_2}{\partial z_1} \ = \ 0 \ ;$$

  this equation admits as solution $\eta_2(z_1, z_2, t) = \xi(z_2 - a_2 t)$, with ξ an arbitrary function.
- Corrections should also be introduced in the formulas relating to Examples 5 & 6; these would require displaying rather large formulas and hence we will just alert the reader about this fact.
- Examples 7 through 10 are (obviously) unaffected.

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As mentioned above, the error in (6) has some more substantial consequence in Section VIII. In fact, the main conclusion reached there turns out to be wrong: for *simple* (deterministic or random) *symmetries*, there is a full equivalence between an Ito and the corresponding Stratonovich equation. In the deterministic case, this was proved by Unal [3]; he also showed that this is not the case for general symmetries: in particular for symmetries acting on time as well, there is an auxiliary condition (amounting to a third order differential equation) to be satisfied; see Proposition 1 in Unal's paper.

Repeating the computation with the correct form of the Ito Laplacian (6), one can prove that δ^i defined in (61) is identically zero. The full computation will be given elsewhere [4], but the one for the scalar case is rather simple. In fact, in this case $\rho = (1/2)\sigma_x \sigma$. Moreover the second determining equation (11) guarantees that $\varphi_w = \varphi \sigma_x - \sigma \varphi_x$; writing φ_{ww} and φ_{wx} as differential consequences of this, and with standard computations, one easily obtains that

$$\delta \ := \ \varphi \, \rho_x \ - \ \rho \, \varphi_x \ - \ (1/2) \, \Delta \varphi \ = \ 0 \ .$$

Correspondingly, the phrase summarizing the results of Section VIII in the Conclusions (Section X), i.e. the paragraph starting with "We have also discussed the relation..." (up to "On the other hand...") is also wrong. A correct version of this statement would read as follows:

"The simple (deterministic or random) symmetries of an Ito equation and those of the corresponding Stratonovich one do coincide".

We apologize to the readers, and thank the anonymous Referee of [4] for pointing out the mistake.

[1] G. Gaeta and F. Spadaro, "Random Lie-point symmetries of stochastic differential equations", *J. Math. Phys.* **58** (2017), 053503 [*arXiv:1705.08873*]

[2] G. Gaeta, "Symmetry of stochastic non-variational differential equations", *Physics Reports* **686** (2017), 1-62 [*arXiv:1706.04897*]

[3] G. Unal, "Symmetries of Ito and Stratonovich Dynamical Systems and Their Conserved Quantities", *Nonlinear Dynamics* **32** (2003), 417-426

[4] G. Gaeta and C. Lunini, "On Lie-point symmetries for Ito stochastic differential equations", to appear in *J. Nonlin. Math. Phys.*