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R. Fürth's 1933 paper "On certain relations between classical statistics and quantum mechanics" ["Über einige Beziehungen zwischen klassischer Statistik und Quantenmechanik", Zeitschrift für Physik, 81 143–162]

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Abstract We present a translation of the 1933 paper by R. Fürth in which a profound analogy between quantum fluctuations and Brownian motion is pointed out. Fürth highlights the existence of uncertainty relations involving the variance of a statistically conserved quantity of a non-equilibrium thermodynamic indicator and the variance of the corresponding current velocity. The phenomenon is entirely classical and traces back to the effect of a fluctuating environment on a measured system. In some sense, Fürth's paper also opened the way to the stochastic methods of quantization developed almost 30 years later by Edward Nelson and others.

Introduction

The thermodynamic uncertainty relations are a remarkable set of inequalities in Stochastic Thermodynamics that bind the coefficient of variation of empirical currents to their averages and to the entropy production rate (see, e.g., the research papers [1,2] and/or the monograph [3] for an overview). In a nutshell, they intimate that achieving currents with a very small coefficient of variation requires, in general, a minimal cost in terms of entropy production. Their name is evocative of the uncertainty relations of quantum physics, which set a bound on the accuracy with which the position of a particle and its velocity can be evaluated.

It is interesting to note that an analogy between the inequality expressing Heisenberg uncertainty relations and a similar one which applies to diffusion processes like Brownian motion was pointed out in a remarkable paper by Reinhold Fürth in 1933. The paper, which points to a profound similarity between the uncertainty arising from quantum fluctuations and that due to random forces acting on a diffusing particle, opened the way in some sense to more recent developments like Nelson's stochastic mechanics approach [4] to quantum mechanics, and the stochastic quantization approach championed by Parisi and Wu [5,6].

We hope to be helpful to the community by providing a translation of this comparatively little known paper. The translation is preceded by a brief biographical sketch of its author and is followed by some remarks on the translation and a brief commentary. In the commentary, we couch Fürth's result in the modern language of stochastic processes and delve into its implications for recent developments in stochastic thermodynamics [3]. Notice that the author's references appear as footnotes as in the original paper. The references due to the curators appear in brackets and are listed at the end.

Reinhold Fürth

Possibly the most detailed information about Fürth's biography comes from his obituary, published in the Year Book of the Royal Society of Edinburgh [7].

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Reinhold Henry Fürth was born on 20 October 1893 in Prague, at that time capital of Bohemia, a province of the Austro-Hungarian empire. He was the only son of professional parents with literary and artistic interests. He studied at the Austrian State *Gymnasium* in Prague where the emphasis was on classical languages. Later, from 1912 to 1916, he attended the Royal and Imperial German Charles Ferdinand University in Prague. There, he took classes in both experimental and theoretical physics as well as in mathematics. Thus, from the very beginning of his studies he manifested interest for all aspects of physics. Fürth joined the Charles Ferdinand University at a time when that institution had possibly just passed its highest point in physics as the home of world-renowned scientists, such as Ernst Mach and Albert Einstein. In [8] Fürth recalls that

> *At that time Einstein had just left Prague to take up his professorship at Zürich. I missed him directly, but everywhere there were traces. For instance, stored in a cupboard was a machine for multiplying electric charges; Einstein had designed it, although he rarely concerned himself with experimental matters. More important, Einstein's successor as professor of theoretical physics, Philipp Frank, was one of the few people who had taken up research on his ideas. He introduced me to relativity theory. Some of my senior colleagues had themselves been pupils of Einstein, so they acquainted me with his methods of teaching and his personality.*

Fürth only met Einstein in person in 1920 at the age of 27. In Prague, Frank drew Fürth's attention to Einstein's theory of Brownian motion. The subject fascinated Fürth so much that he soon started research work in the field. He published several peer-reviewed related articles including the monograph [9] on fluctuation phenomena in physics. He later contacted Einstein to ask permission to edit the collection of his papers on Brownian motion [10]. Fürth's name is now well-known also because of the English translation [11] of this work which is widely read even now.

At the beginning of twentieth century the identification of noise with irregular fluctuations was not yet taken for granted. Furthermore, noise measurement protocols still partially relied on human sensitivity rather than on fully automatic indicators from which observers could take directly numerical outcomes. In 1922 Fürth published the paper [12] where he managed to explain the variation of the recorded charge of the electron in shot noise measurements based on the "physiological" nature of the so-called "ear balancing" experiment protocol. This paper nowadays represents a landmark in the history of the development of measurement techniques of disturbances in early twentieth-century telephone and radio engineering [13].

Fürth took his doctorate in 1916 with a thesis based on an experimental investigation of critical opalescence in binary liquid mixtures. In 1920 he qualified as *Privatdozent*.

In 1927 he became *Aussenordentlicher Professor* in theoretical physics and in 1931 full professor of experimental physics and Head of the physics department, always in his home university. In 1933, the same year of publication of the article we are translating here, he put forward the idea of stars made-up of anti-particles [14], only two years after Dirac introduced the idea of anti-proton [15]. We refer to [16], especially chapters 5 & 10, for a history of the introduction and initial development of the idea of anti-matter.

In 1937 Fürth was appointed Dean of the Faculty of Science, a position which he could only maintain until 1938. In the fall of that year, as an indirect consequence of the "Munich agreement" which under the misplaced hope of an appeasement provided the cession to Nazi Germany of part of the Czechoslovak Republic, Fürth was forced to resign his positions to permit the appointment of "aryans" in his positions. In the spring 1939 Nazi Germany occupied Bohemia and Moravia and a campaign of persecution against Jews and Czech public figures started. Fürth was dismissed from university. Together with his wife he managed to escape from Czechoslovakia just before the outbreak of World War II as well as to salvage a considerable part of their possession [7]. Fürth emigrated to Edinburgh following an invitation of Max Born who in 1936 had assumed there the Tait chair of Natural Philosophy. Fürth remained in Edinburgh for eight years. Initially, he was supported by a scholarship from the Society for the Protection of Science and Learning. Later, he held the Dewar Research Fellowship and from 1942 a part-time lectureship. Fürth took a house at 60 Grange Loan near that of Born at 84. Fürth closely collaborated with Born for all the period of his stay in the capital of Scotland. In [17] pag. 40, Born puts Fürth in first place among his capable collaborators in Edinburgh:

> [Fürth] *who came as a refugee just before the beginning of the war ... was a great help to me in directing the work of my pupils in the thermodynamics of crystals and other topics.*

Robert William Pringle was among these pupils. Pringle became in 1956 the founder of Nuclear Enterprises(GB)Ltd, one of the largest companies in the world specialised in the field of nucleonics and ultrasonic diagnostic equipment [18]. Pringle's Ph.D. project focused on the development of computational methods of Fourier transforms whose calculation was a long and tedious undertaking at that time. As Pringle had a preference for experimentation, Fürth joined the supervision. Together Born, Fürth and Pringle devised an analogue computer for the calculation of Fourier transforms, employing one of the first photomultipliers. Their first model was constructed in the Mathematical Physics Department at Edinburgh. It was described in a joint paper in Nature [19], and in a subsequent patent application. An engineered version was built by Ferranti in 1946–47 and generated considerable interest. In the same period Fürth constructed other two technically innovative devices: a microphonometer on a tuning fork movement and a cathode ray oscillograph display that was capable of a continuous magnification upto to 200 times or more [8]. All of this he managed to achieve only supported by very slender budgets.

In a recognition of his achievements Fürth was elected fellow of the Royal Society in 1943. In 1947 he and his wife became naturalized British subjects. In the same year he was also appointed Reader in theoretical physics at Birkbeck college in London where he remained until retirement in 1961.

Notable works of that period are [20, 21] which can be considered as the earliest examples of the approach known as sociophysics.

There, Fürth criticizes the use of purely mechanistic approaches to social phenomena because [21]

what the sociologist or politician means by "force" ... can neither be defined mathematically nor measured quantitatively and there is therefore no justification for the assumption that a superposition of various such "forces" can ever lead to their mutual compensation. Besides, there is no reason that this would result in an equilibrium.

He also points out that the inadequacy of purely statistical models based on chance games:

there is no justification that the fate of an individual is subject to fixed probabilities nor that it is independent of the behaviour of other individuals within the same community at the same time or some previous time.

Instead he proposes

adopting a model incorporating both the causal and the chance aspect. In physics such unification has been brought about during the last half century by the development of "statistical mechanics" .

Fürth argues that social communities can be conceptualized as large assemblies of distinct and to some extent independent units yet strongly influencing each other and thus in analogy with co-operative phenomena in physics described by statistical mechanics. Fürth contribution had immediate resonance see, e.g. [22].

For his work on the Statistical Thermodynamics of Liquids and in recognition of many valuable contributions to statistical physics, Furth was awarded the Keith medal of the Royal Society of Edinburgh in 1965. His overall scientific production amounts to some 200 papers and includes several books and patents. He is also remembered for his great love of music and wide cultural interests. He died in 1979.

Text

From the Physics Institute of the German University in Prague

On certain relations between classical statistics and quantum mechanics.

by Reinhold Fürth in Prague.

With 4 figures. (Received on January 19, 1933.)

Abstract

We highlight the formal analogy between the differential equations for the probability distribution of the position of a mechanical system according to classical statistics, and those according to quantum mechanics: equations that can also be interpreted as describing the motion of a cluster of identical particles, i.e., a diffusion. The physical origin of such a diffusion is ascribed in the classical case to the collision with molecules in the surrounding matter, and, in the case of quantum mechanics, to the uncertainty relations. In the latter case, diffusion in the absence of forces is discussed and a simple derivation of the uncertainty relations is given on this basis. This line of reasoning can be carried over to classical diffusion, allowing the derivation of an inequality for the variance of the position and velocity, in strict analogy with Heisenberg's uncertainty relations. The relation thus found can also be applied to a single particle and, more generally, to an arbitrary mechanical system, since it states that the simultaneous measurement of the position and corresponding velocity is possible only up to a maximal accuracy as a consequence of the Brownian motion. We discuss the relation of this finding with the problem of determining the accuracy of measuring a physical quantity with a mechanical measurement device, and obtain the result that also in this case there exists, in analogy with quantum mechanics, an accuracy limit which cannot be overcome. Finally, we clarify, from the point of view of wave mechanics, why the classical diffusion equation holds for a real density function with a real diffusion coefficient, in contrast to the Schrödinger equation, which holds for a complex function with an imaginary coefficient. We also show how this is related to the observability of physical quantities and to the reversibility versus irreversibility of natural processes.

We present in what follows a discussion of certain relations between, on the one hand, classical statistics (classical diffusion theory and the theory of B r o w nian motion) and, on the other hand, quantum mechanics. This discussion arises from formal considerations and, to the best of my knowledge, has not yet been addressed in this context, although it might be already known to some. It is possible to show, in particular, that H e i s e n b e r g's uncertainty relations carry over to processes which are governed by classical statistics and that thus it is possible to bring about new perspectives on the often addressed question of the limit of measurability with a measurement device. Moreover, we attempt to make the physical meaning of the above-mentioned similarities and differences more precise.

1

The classical theory of diffusion is governed[1] by the generalised diffusion equation

$$\frac{\partial u}{\partial t} = D \cdot \varDelta u - \operatorname{div}(u\mathfrak{v}) \tag{1}$$

where $u(x, y, z, t)$ denotes the density as a function of the position and time, D (assumed constant) the diffusion coefficient and $\mathfrak{v}$ the velocity vector of the convection current occasioned by external forces. The solution of this equation under given boundary conditions determines the distribution of the density at any future instant of time, if the distribution is known in the present.

If one interprets the diffusion experiment as a collective experiment with a spatial ensemble of many identical particles, then $u\,\mathrm{d}V$ is the relative frequency with which any element of the ensemble is found in the volume element $\mathrm{d}V$ at time t during the collective experiment, provided u satisfies, for all t, the normalization condition

$$\iiint u\,\mathrm{d}V = 1. \tag{2}$$

The replacement of the spatial ensemble with a virtual ensemble turns the diffusion Eq. (1) into an equation for the "probability density" u of the position of an individual particle, that can be computed as a function of time if it is known at time zero: namely, [the replacement turns (1)] into S m o l u c h o w s k i's differential equation for the B r o w nian motion of an individual particle under the action of external forces[2].

It is possible to show that S m o l u c h o w s k i's equation is a special case of another differential equation that can be derived under very general conditions for the B r o w nian motion of an arbitrary mechanical system and that is usually referred to as the F o k k e r - P l a n c k differential equation[3]. Following S c h r ö d i n g e r[4], this equation can be written as

$$\frac{\partial u}{\partial t} = Fu \tag{3}$$

where F denotes a certain differential operator, which, in agreement with (1), reduces to $F = D\varDelta\ - \operatorname{div}\mathfrak{v}$ in the case when the system is a particle under the action of a force.

The differential Eq. (3) is, as S c h r ö d i n g e r[5] also already pointed out, formally identical to the time dependent S c h r ö d i n g e r differential equation of wave mechanics for the wave function ψ which is usually written in the form

$$-\frac{h}{2\pi \mathrm{i}}\frac{\partial \psi}{\partial t} = H\,\psi \tag{4}$$

where H denotes the H a m i l t o n operator for the mechanical problem of interest.

According to the statistical formulation of wave mechanics, this equation is also a "probability equation", inasmuch as it allows one to compute, from the knowledge of $\psi(q)$ at time zero, the same quantity at any arbitrary later instant of time. The "probability amplitude" ψ is linked to the probability density for the location of the system in a certain volume element of the q-space by the relation

$$w = \psi\psi^* \tag{5}$$

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[1] See, e.g., Frank-Mises, *Differential- u. Integralgleichungen d. math. Physik* 2, 248.

[2] M. v. Smoluchowski, *Ann. d. Phys.* 43, 1105, 1915.

[3] See, among others, F. Zernike, *Handb. d. Phys. Bd. III*, S. 457.

[4] E. Schrödinger, *Ann. de l'Inst. H. Poincaré* 1931, S. 296ff. *S. Ber. Berl. Akad.* 1931, S. 148; see also J. Metadier, *C. R.* **193,** 1173, 1931.

[5] E. Schrödinger, *Ann. de l'Inst. H. Poincaré* 1931, S. 296ff. *S. Ber. Berl. Akad.* 1931, S. 148; see also J. Metadier, *C. R.* **193,** 1173, 1931. [sic].

(ψ^* is the complex conjugate of ψ) provided that ψ satisfies the normalization condition

$$\iiint \psi\psi^* \,\mathrm{d}V = 1 \tag{6}$$

By reversing this line of reasoning, one can also build the quantity w defined via (5) as the phase point density of a large number of identical, non-interacting systems in q-space. Equation (4) then determines the evolution of this distribution density and allows the computation of the density at any further time, if the density function is assigned at time zero.

In the special case of a point mass m being subject to the action of a force which can be derived from a potential U, Eq. (4) reads

$$-\frac{h}{2\,\pi\,i}\frac{\partial\psi}{\partial t} = -\frac{h^2}{8\,\pi^2\,m}\Delta\psi + U\,\psi \tag{7}$$

The discussion of this equation teaches us, as Ehrenfest first showed[6], that when the assigned forces act on a particle, the centre of mass of a cluster of particles obeying the conditions mentioned above moves in the usual three-dimensional space according to the prescriptions of classical mechanics, and also that the cluster of particles spreads around the centre of mass via a sort of diffusion. We therefore encounter here a convection current overlaid with a diffusion, in analogy with the motion of a cluster of particles according to the classical diffusion Eq. (1).

As we are interested only in the last phenomenon, we wish to set the external force to zero in what follows. Equations (1) and (7) then become formally identical, namely

$$\frac{\partial u}{\partial t} = D\,\Delta\,u \tag{8}$$

and

$$\frac{\partial\psi}{\partial t} = \varepsilon\Delta\,\psi \tag{9}$$

where we use the shorthand

$$\varepsilon = \frac{i\,h}{4\,\pi\,m} \tag{10}$$

When subject to the same boundary and initial conditions, the solutions of (8) and (9) are hence completely the same. Sure enough, a substantial difference arises from the fact that in the case of quantum mechanics, it is not the function (in general complex) ψ, but rather, according to (5), its norm that plays the role of the density function and, according to (10) the diffusion coefficient is purely imaginary. We return to the physical meaning of this fact later.

2

The deeper reason for the analogy emerging from the comparison in Sect. 1 of the motion of a cluster of particles according to the classical theory of diffusion and that according to quantum mechanics resides in the fact that in both cases the velocities of individual particles in the cluster differ and obey a statistical law.

In the former case, this phenomenon stems from the fact that particles undergo irregular collisions with molecules in the surrounding environment, whereby their momentum continuously varies in intensity and direction in such a way that there is no relation between the changes in momenta of distinct particles. This becomes manifest when considering an individual particle in an irregular Brownian motion. For a particle cluster, we see this in the fact that for an assigned initial state of the cluster and initially vanishing "macroscopically" measured velocity, the particles actually possess velocities that are irregularly distributed across the cluster, and that over the course of time the initial distribution varies in the characteristic way of a diffusion.

In the case of quantum mechanics, the very assumption of an initial density distribution implies that the condition of vanishing initial velocity of all the particles cannot be strictly satisfied. According to Heisenberg's fundamental uncertainty relations governing quantum mechanics, a complete assignment of the initial velocity of the particles would be possible only in the presence of a complete uncertainty about the initial positions. As some information about the initial position of the particles is conveyed by the assignment of an initial distribution, one must admit a certain blurring of the initial velocities, i.e., a certain statistical distribution of the initial velocities

[6] P. Ehrenfest, *ZS. f. Phys.* **45**, 455, 1927.

of the cluster particles. It necessarily follows that after a certain time, a variation of the initial density distribution, i.e., a "diffusion", of the cluster must have occurred.

That the uncertainty on the value of position of the particles of the diffusing cluster really satisfies H e i s e n b e r g's uncertainty relations, with the uncertainty about the value of the velocity (momentum), has been shown by H e i s e n b e r g[7] and K e n n a r d[8] among others. A brief derivation may be given here for the one-dimensional case, which, without resorting to the theory of transformations, makes use[9] only of equation (9) and its complex conjugate that in one dimension takes the form

$$\left.\begin{aligned} \frac{\partial \psi}{\partial t} &= \varepsilon \frac{\partial^2 \psi}{\partial x^2} \\ \frac{\partial \psi^*}{\partial t} &= -\varepsilon \frac{\partial^2 \psi^*}{\partial x^2} \end{aligned}\right\} \tag{11}$$

Let x_0 be the initial position of one particle in the cluster, v its initial velocity and x its position after time t, then

$$x = x_0 + v\,t \tag{12}$$

holds. If the centre of mass at time zero is located in the origin of the coordinates and its velocity is zero, i.e. $x_0 = 0$ and $\bar{v} = 0$, then according to (12) it is also clear that $\bar{x} = 0$ for all t. Upon evaluating the expectation value of the square of (12), one gets

$$\overline{x^2} = \overline{x_0^2} + 2\,\overline{x_0\,v}\,t + \overline{v^2}\,t^2 \tag{13}$$

By definition

$$\overline{x^2} = \int_{-\infty}^{+\infty} x^2\,\psi\psi^*\,\mathrm{d}x \tag{14}$$

holds true. Using Eq. (11) and under the assumption that ψ vanishes sufficiently fast at infinity, after a simple calculation with the help of (14) one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{x^2} = 2\,\varepsilon \int_{-\infty}^{\infty} x \left(\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x}\right) \mathrm{d}x \tag{15}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\overline{x^2} = -8\,\varepsilon^2 \int_{-\infty}^{\infty} \frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x}\mathrm{d}x \tag{16}$$

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\overline{x^2} = 0 \tag{17}$$

It follows from (17) that $\overline{x^2}$ must be a quadratic function of time that agrees with (13); it also follows from (16) that $\overline{v^2}$ as coefficient of t^2 in (13) satisfies

$$\overline{v^2} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\overline{x^2} = -\,4\,\varepsilon^2 \int_{-\infty}^{\infty} \left|\frac{\partial \psi}{\partial x}\right|^2 \mathrm{d}x \tag{18}$$

According to H e i s e n b e r g[10], it now follows from the self-evident inequality (19)

$$\left|\frac{x}{2\,\overline{x^2}}\,\psi + \frac{\partial \psi}{\partial x}\right|^2 \geqq 0 \tag{19}$$

[7] W. Heisenberg, *ZS. f. Phys.* **43,** 172, 1927.
[8] E. H. Kennard, loc. cit. **44,** 326, 1927.
[9] Here I need to thank Mr. K. L ö w n e r, Prague, for some hints.
[10] W. H e i s e n b e r g, *Die physikalischen Prinzipien der Quantentheorie.* Leipzig 1930. Page 13 and following.

with use of (6) and (14) that

$$\int_{-\infty}^{\infty} \left|\frac{\partial \psi}{\partial x}\right|^2 \mathrm{d}x \geqq \frac{1}{4\,\overline{x^2}}$$

whence from (18)[11]

$$\overline{x^2}\;\overline{v^2} \geqq -\varepsilon^2 \tag{20}$$

If one introduces uncertainty to the position and momentum of the particle cluster under consideration by means of the relations

$$\left.\begin{aligned} \mathit{\Delta} x &= \sqrt{\overline{x^2}} \\ \mathit{\Delta} p &= m\sqrt{\overline{v^2}} \end{aligned}\right\} \tag{21}$$

combining (20) with (10), their product follows the Heisenberg relation

$$\mathit{\Delta} x\, \mathit{\Delta} p \geqq \frac{h}{4\,\pi} \tag{22}$$

Equality holds here if and only if the inequality (19) holds as an equality. Integrating the latter yields for ψ

$$\psi = \text{Const.}\, e^{-x^2/4\,(\mathit{\Delta} x)^2} \tag{23}$$

and, for the density of the particle cluster (5), in consideration of (6) yields the Gaussian distribution

$$w = \frac{1}{\sqrt{2\,\pi}\mathit{\Delta} x}\, e^{-x^2/2\,(\mathit{\Delta} x)^2} \tag{24}$$

If ψ takes the special form (23) at time $t = 0$ then it follows from (15) that $\frac{\mathrm{d}}{\mathrm{d}t}\overline{x^2} = 0$ and, as a consequence, the coefficient of t disappears from (13). If there is a corresponding initial distribution of the position in the particle cluster under consideration, so that $(\overline{x_0 v}) = 0$ also holds at the same time, the variance of the positions and the initial velocities of the individual[12] particles are also statistically independent from one another. Conversely, by no means does the existence at time zero of the density distribution (24) imply by itself the statistical independence of position and velocity and hence also the vanishing of the linear term in (13).

3

As per the beginning of § 1, it is natural to apply the above reasoning, based on the Heisenberg uncertainty relation in the quantum mechanical case, to the case of classic diffusion. In this case too, we restrict ourselves to the one dimensional case with vanishing convection current. We start from equation (8), which in one dimension reads

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \tag{25}$$

where, in agreement with (2), u satisfies the condition

$$\int_{-\infty}^{\infty} u\,\mathrm{d}x = 1 \tag{26}$$

We define the uncertainty of the particle cluster by means of the quantity $\overline{x^2}$ as

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2\, u\,\mathrm{d}x \tag{27}$$

[11] In the German original the reference is to (17) which should be a typo as pointed out to us by the referee.
[12] **TN**: the original text reads “single” particle which is probably a typo.

At $t = 0$, the centre of mass of the cluster lies again at the origin of the coordinates so that $\overline{x_0} = 0$. To start with, we look for the derivation of the analog of Eq. (13), which expresses how the uncertainty $\overline{x^2}$ initially present in the diffusing particle cluster grows over the course of time. Upon using Eq. (25) and the assumption that u vanishes sufficiently fast at infinity, we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\overline{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} x\,u\,\mathrm{d}x = \int_{-\infty}^{\infty} x\,\frac{\partial u}{\partial t}\,\mathrm{d}x \\ &= D\int_{-\infty}^{\infty} x\,\frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x = 0\end{aligned}$$

Thus, the center of mass of the cluster remains at rest, as straightforwardly implied by the absence of a convection, so that for all times $\overline{x} = 0$.

In an analogous way, it follows from (27) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{x^2} = \int_{-\infty}^{\infty} x^2\,\frac{\partial u}{\partial t}\,\mathrm{d}x = D\int_{-\infty}^{\infty} x^2\,\frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x = 2\,D \tag{28}$$

and therefore that $\overline{x^2}$ is a linear function of time of the form

$$\overline{x^2} = \overline{x_0^2} + 2\,D\,t \tag{29}$$

The comparison of (29) with (13) shows that in both cases the uncertainty over the position grows indefinitely over a sufficiently long time and thus that a diffusion of the cluster occurs. Here the growth of $\overline{x^2}$ occurs independently of $\overline{x_0^2}$ and *linearly* in time, whereas before the growth in time is *quadratic*, and because of (20), is itself dependent upon $\overline{x_0^2}$ (it takes place in a particularly sudden way if $\overline{x_0^2} = 0$ inasmuch as $\overline{v^2}$ becomes infinitely large). Finally, if the linear term in t is non-vanishing, so that the dispersion of the positions and of the velocities are not statistically independent at time zero, it may be that the cluster first contracts to a minimum and only afterwards spreads out.

The formal causes for these differences have already been discussed at the end of Sect. 1. They are physically explained by the fact that in the case of classical diffusion there is no "initial velocity" of the particles and therefore no equation of the form (12) exists. Furthermore, the instantaneous velocity of the particles is due to collisions with molecules in the surrounding environment, as already mentioned[13]. On the basis of the statistical independence of the dispersion process and of the initial distribution in the classic case, one can immediately write down Eq. (29). Indeed, this equation expresses that the dispersion is due to two causes: the initial dispersion and the diffusion. This corresponds to the fact that the "mean squared error" of x is the sum of these two ingredients, of which the latter is the well-known E i n s t e i n law for the mean squared error of B r o w nian motion.

In order to find the analog of the uncertainty relations (20) to (22), we first need to find a suitable definition of velocity for the classical diffusion. From the above it is clear that this role can by no means be played by the microscopic velocity produced by molecular collisions. Likewise the macroscopic velocity of the cluster regarded as a single entity, or, strictly speaking, the velocity of its centre of mass, is not a good candidate since it vanishes. A suitable quantity comes from the consideration of the "diffusion current", i.e., the quantity of diffusing matter that passes through a given unit area in the diffusion domain in the unit of time. As it is well known,[14] the vector $\mathfrak{Q}$ of the diffusion current is a local function in the diffusion domain, connected to the scalar u by the relation

$$\mathfrak{Q} = -D\,\mathrm{grad}\,u \tag{31}$$

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[13] If one, following S c h r ö d i n g e r (Ber. Ber. 1930, S. 296, Nr. 19), chooses the value of $\overline{x_0^2}$ such that Δx is as small as possible and that the product $\Delta x \Delta p = \frac{h}{4\pi}$ holds true such that the initial distribution is given by (23), then in (13) the second term of the right hand side vanishes, and the first and the third become equal upon setting $\overline{v^2} = -\frac{\varepsilon^2}{\overline{x^2}}$. This implies that for $\overline{x^2}$

$$\overline{x^2} = \frac{h}{2\,\pi\,m}t \tag{30}$$

In this case, the formal analogy with (29) becomes strikingly evident by replacing D with the absolute value of the "imaginary diffusion coefficient" ε according to (10).

[14] Compare ref. 1.

Based on the fact that u is nothing else than the density of the diffusing matter, we find the corresponding velocity vector $\mathfrak{v}$ according to

$$\mathfrak{v} = \frac{1}{u}\mathfrak{Q} = -D\frac{1}{u}\,\mathrm{grad}\,u \tag{32}$$

which in the one-dimensional case becomes

$$v = -D\frac{1}{u}\frac{\partial u}{\partial x} \tag{33}$$

If we now compute the mean value of v for the particle cluster at a certain time instant, we obtain, by definition, using (25)

$$\overline{v} = \int_{-\infty}^{\infty} v\,u\,\mathrm{d}x = -D\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}\mathrm{d}x = 0$$

as it must be, since $\overline{v}$ is the macroscopic velocity of the centre of mass.

For the mean value of $\overline{v^2}$, one finds

$$\overline{v^2} = \int_{-\infty}^{\infty} v^2\,u\,\mathrm{d}x = D^2\int_{-\infty}^{\infty}\frac{1}{u}\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x \tag{34}$$

By a straightforward application of the reasoning in Sect. 2 one can establish an inequality for the product $\overline{v^2}\;\overline{x^2}$, by proceeding once again from the self-evident inequality

$$\left(\frac{1}{u}\frac{\partial u}{\partial x} + \frac{x}{\overline{x^2}}\right)^2 \geqq 0 \tag{35}$$

whence by expanding the product, it follows

$$\frac{1}{u}\left(\frac{\partial u}{\partial x}\right)^2 \geqq -2\frac{x}{\overline{x^2}}\frac{\partial u}{\partial x} - \frac{x^2\,u}{(\overline{x^2})^2}$$

Upon integrating, a simple calculation, making use of (26) and (27), yields

$$\int_{-\infty}^{\infty}\frac{1}{u}\left(\frac{\partial u}{\partial x}\right)^2 \geqq \frac{1}{\overline{x^2}}$$

whence finally according to (34)

$$\overline{x^2}\;\overline{v^2} \geqq D^2 \tag{36}$$

As one can see, the inequality (36) has the same form of the inequality (20), which turns into (36) if one again replaces the absolute value of ε with D. Introducing the notation Δx and Δv in analogy with (21), we write our uncertainty relation in the simpler form

$$\Delta x\,\Delta v \geqq D \tag{37}$$

stating that in a classically diffusing particle cluster, the position and the velocity of the particles at any instant of time cannot be *simultaneously* determined with arbitrary accuracy and furthermore, that the product of the uncertainties must always be larger than the diffusion coefficient D.

The lower bound is attained, i.e., the inequality turns into an equality, if and only if (35) also holds as an equality. The solution of the differential equation obtained in this way immediately yields

$$u = \frac{1}{\sqrt{2\,\pi}\Delta x}\,e^{-x^2/2(\Delta x)^2} \tag{38}$$

having taken (26) into account, which is again the G a u s sian distribution, as in the quantum mechanical case, in formal agreement with (24).

Whereas in the present case the equality $\Delta x\,\Delta v = D$[15] necessarily follows from the occurrence of the distribution (38), before, the occurrence of the distribution (24) was only a necessary but not sufficient condition for the product $\Delta x\,\Delta v$ to attain its minimum. Furthermore, whereas in a cluster of particles left to itself and satisfying at time zero the minimum uncertainty condition this condition continues to hold at any further time (because the distribution (38) is self-sustaining), in the quantum mechanical case the minimum condition is only instantaneously satisfied, e.g., at time zero, and not again later (since the form of the distribution (23) is not preserved by the motion of the particles). Finally, it should be emphasized that in the classical case one can always think of a cluster of particles satisfying the minimum condition as being brought about by the diffusion of a cluster which at a certain instant of time was completely concentrated at the origin of the coordinates. In order to see this, one needs only to substitute (29) in (38), and insert the shorthand $\overline{x_0^2} = 2\,D\,t_0$ obtaining

$$u = \frac{1}{2\sqrt{\pi\,D\,(t+t_0)}}\,e^{-x^2/4\,D\,(t+t_0)} \tag{39}$$

which entails that indeed, for $t = -t_0$, u vanishes in the full space except for $x = 0$. In the quantum mechanical case, this reduction, as we have already seen, is not possible.

4

In the two preceding paragraphs we discussed the application of uncertainty relations to a spatial aggregate of identical particles in the quantum and classical cases. As it is well known, the fundamental significance of the uncertainty relation in Quantum Mechanics appears when it is applied to an individual system. It shows that the simultaneous measurement of the position and the momentum of a force-free particle can be performed with a maximum accuracy $h/4\pi$ as predicted by formula (22), since the measurement process of one of the two quantities disturbs the other by an amount such that the product of the uncertainties of both quantities cannot be smaller than this value. One can reformulate the statement for a general mechanical system by saying that the simultaneous measurement of a coordinate q and the impulse canonically conjugated to it is only possible with an uncertainty of the order of magnitude of h.

We can now also apply the relation (37) obtained in Sect. 1, in a straightforward way to the problem of the simultaneous measurement of the position and velocity of a particle which is under the action of irregular impacts and therefore performs a B r o w nian motion. Our relation implies that the product of the uncertainty of a simultaneous measurement of the position and velocity cannot be lower than the value D, where the velocity must be understood as the macroscopic velocity of the particle, i.e., the quantity $\delta x/\delta t$ (assuming that δt is large compared to the time between two successive molecular collisions of the particle). One sees that, as in the quantum mechanical case, there is an actual impossibility of a simultaneous, precise measurement of both position and velocity, which is not, as in Quantum Mechanics, determined by the process of measurement itself and governed by a universal constant, but rather caused by the influence of the environment on the observed system. Consequently, it is clearly not of universal nature (for example, the effect can be made arbitrarily small by lowering the temperature, which determines the value D).

The following argument shows that Eq. (37) also holds true in the case of the measurement of an individual particle: we consider a force-free particle which is located at the origin of the coordinates at time zero and has vanishing macroscopic velocity. If we measure the position of the particle after a short time t, then the expected value $\overline{x^2}$ satisfies Einstein's equation

$$\overline{x^2} = 2\,D\,t \tag{40}$$

from which it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\overline{x^2}}{2} = D$$

If we now exchange the order between time differentiation and averaging, we get[16]

$$\overline{\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{x^2}{2}\right)} = \overline{\left(x\frac{\mathrm{d}}{\mathrm{d}t}x\right)} = D \tag{41}$$

[15] **TN**: the German original text reads here $\Delta x\,\Delta v = 0$ that is clearly a typo.

[16] **TN**: the stochastic differential here must be interpreted in Stratonovich sense.

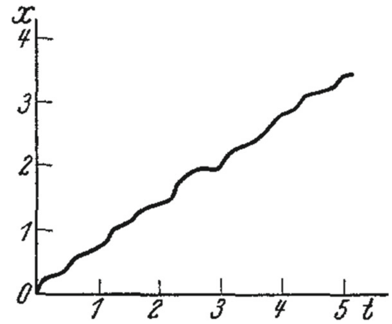

Fig. 1 Plot of the position of a B r o w nian particle as a function of time (schematic drawing)

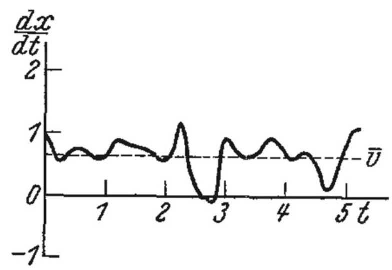

Fig. 2 Velocity v of the particle as a function of time computed from Fig. 1 (dashed line: mean value $\overline{v}$)

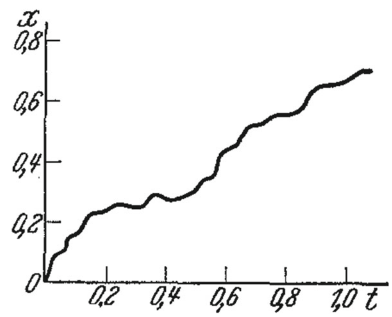

Fig. 3 Five-times magnification of the beginning of the plot in Fig. 1 (schematic drawing)

Now x is evidently the uncertainty of the position of the particle (we assumed $x = 0$ at time zero) caused by the B r o w nian motion, and similarly $\mathrm{d}x/\mathrm{d}t$ is the uncertainty over the velocity (which we assumed to be vanishing at time zero) brought about by the same causes. The product $\overline{\left(x\frac{\mathrm{d}}{\mathrm{d}t}x\right)}$ thus specifies the expected value of averaging over many measurements of the uncertainty product $\Delta x \Delta v$, which according to Eq. (41) is equal to D. The fact that we obtained exactly the lower bound instead of the inequality (37) is due to the fact that we evaluated the mean over repeated measurements of a particle, which was assumed to always have the same starting position and velocity. It is immediately obvious that without this assumption the uncertainty would only increase, so that the product $\Delta x \Delta v$ is actually larger than D, as required by the relation (37).

Our relation states that an increase in the measurement accuracy of the position of a B r o w nian particle reduces the accuracy of a simultaneous measurement of its velocity and vice versa. The physical meaning of this statement can be visualized with the help of the following Figs. 1, 2, 3 and 4: Fig. 1 plots the position $x(t)$ as a function of time of a particle falling under gravity in a liquid, observed with a certain magnification; and Fig. 2 represents the function $v(t) = \dot{x}(t)$ from the particle in Fig. 1. Figure 3 shows the beginning of Fig. 1, plotted with a stronger magnification, and Fig. 4 shows the corresponding velocity curve.

One can immediately see how increasing the accuracy of determining the position by increasing the magnification necessarily increases the uncertainty in the velocity. Our relation expresses a fact known to anyone familiar with B r o w nian motion in an exact way: that the trajectory of a B r o w nian particle exhibits more discontinuities with increased magnification.

As in the case of Quantum Mechanics, we can extend the uncertainty relation (37) to general mechanical systems of any kind that are in contact with a surrounding "temperature bath". Then, every degree of freedom is associated with the B r o w nian motion of the corresponding coordinate, which we denote again by x. The

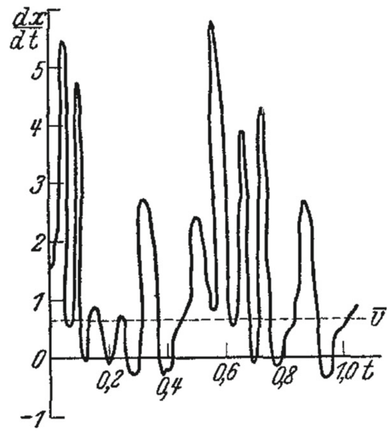

Fig. 4 Velocity v computed from Fig. 3 (dashed line: mean value $\overline{v}$)

F o k k e r - P l a n c k Eq. (3) takes the place of the differential Eq. (25) or (8). It is plausible also in this general case that an uncertainty relation of the form

$$\Delta x\, \Delta v \approx D \tag{42}$$

holds true, where v is the velocity associated to the coordinate x, and D denotes the coefficient of the term $\frac{\partial^2 u}{\partial x^2}$ on the right-hand side of (3) and expresses the characteristic constant of this B r o w nian motion. The relation (42) states that the simultaneous measurement of the coordinate x and of its associated velocity v is only possible with an uncertainty of order D.

5

We can extend the domain of validity of Eq. (42) to any non-mechanical quantity, since any physical quantity, even of non-mechanical nature, is measured using mechanical measurement instruments. For example current is measured using a galvanometer, which consists of mechanical components. We assume that the "deflection" x of the mechanical instrument is proportional to the quantity J to be measured (for example, the deflection of a galvanometer is proportional to the intensity of the current). When this is not the case from the start, one can always apply a compensation method in order to implement the desired condition within narrow limits. Let $\dot{J}$ be the time rate of variation of J. Then, it is true that

$$J = a\,x, \qquad \dot{J} = a\,\dot{x} = a\,v,$$
$$\Delta J = a\,\Delta x, \qquad \Delta\dot{J} = a\,\Delta\dot{x} = a\,\Delta v$$

whence with the help of (42)

$$\Delta J\, \Delta\dot{J} \approx a^2\, D \tag{43}$$

The relation (43) shows that, although one can arbitrarily increase the measurement accuracy by choosing an appropriate measurement device specifically by reducing a, simply increasing the reading accuracy of the pointer cannot improve the precision of a simultaneous measurement of the quantity J and of its time rate of variation above a certain value, because of the B r o w nian motion of the measuring instrument. One can thus reduce a by reinforcing the magnetic field in a moving coil galvanometer with given mechanical properties and, as a consequence, enhance the accuracy of current measurement, at least in principle, arbitrarily: one cannot, however, achieve any reduction of the product $\Delta J\, \Delta\dot{J}$ by a simple increase of the reading accuracy, for example by magnifying the deflection using a microscopic reading pointer[17], using a thermal relay[18] or a light electric relay.[19]

The problem of the limits of measurement accuracy due to B r o w nian motion of instruments, in particular, of galvanometers, has been recently repeatedly discussed by several authors[20], and it has been thoroughly debated

[17] G. Ising, *Ann. d. Phys.* **14** 755, 1932.

[18] N. Moll and N. Burger, *Phil. Mag.* **1,** 624, 1925.

[19] L. Bergmann, *Phys,ZS.* **32,** 688, 1931.

[20] G. Ising, *Phil. Mag.* **1,** 827, 1926; *Ann. d. Phys.* **8,** 911, 1931; **14,** 755, 1932; F. Zernike, *ZS. f. Phys.* **40,** 628, 1926; **79,** 516, 1932; R. Gans, *Schriften d. Königsberger Gel. Ges.* **7,** 177, 1930; M. Czerny, *Ann. d. Phys.* 12, 993, 1932.

by which procedures one can perform the most accurate possible measurement of a quantity of interest with an instrument of a given type. In my opinion, these discussions have always overlooked an important point. The task of the experimentalist is certainly that of recording the quantity J of interest as a function of time, i.e., the function $J(t)$, with the highest accuracy possible. If one restricts one's attention to a short interval of time, this requirement is equivalent to the task of *determining a quantity* J *and its variation time rate* $\dot{J}$ *at a given instant of time with the highest possible accuracy*. The relation (43) shows that with a given instrument this is possible only with an uncertainty that is completely independent of any procedure which increases the reading accuracy of the pointer.

The procedures suggested by many authors to increase the measurement accuracy of J despite the B r o w nian motion which are taking many readings and computing their average (which should be then more precise than an individual measurement), or using an integrating measuring instrument, make sense only when one knows in advance that the quantity of interest is exactly constant. But how can one know this without having first performed a corresponding measurement to ascertain such a stipulation? If one really tried this, then one would obtain, by repeated observation or by continuous recording, a time dependence of the pointer deflection (because of the B r o w nian motion), from which it would certainly not be possible to determine whether the observed quantity remained constant, or whether it varied in time within the limit of accuracy of the recorded fluctuations. This vicious circle is the reason why the methods proposed to increase measurement accuracy are not really feasible.

We can actually say with certainty that the requirement of constancy of J implied by the mentioned procedure is certainly not satisfied, because any macroscopically defined quantity, which can be measured by a macroscopic measurement instrument, undergoes fluctuations. For instance, in reality there is certainly no constant electromotive force, even if the power source is protected from external interference with all possible refinement, because of the occurrence of spontaneous fluctuations of the potential induced by the thermal motion of electrons, as has been experimentally shown by several researchers over the last years[21]. Thus, measuring an electromotive force with the highest possible accuracy obviously requires recording its time dependence as precisely as possible, or simultaneously measuring the electromotive force and its variation velocity in a short time interval. But, as we have shown above, this accuracy has an upper limit that is independent of the way the measurement is performed, as a consequence of B r o w nian motion.

6

The results reported in the previous paragraphs are, as it has been repeatedly mentioned, due to the formal analogy between the fundamental differential equations of classical diffusion theory and quantum mechanics, a fact which becomes particularly evident when contrasting Eqs. (8) and (9) in Sect. 1. Already there we have pointed out essential formal differences between the two equations. We now want to try to understand the physical origins of these differences. The following considerations should at the same time contribute to clarify certain ambiguities, which have recently been highlighted by E h r e n f e s t, while giving an invitation to physicists to tackle these problems.

Classical diffusion can be regarded as a current, which, as we saw in Sect. 1, is governed by a differential equation of the form (3), where F is a real differential operator and u is a real function of position and time representing the density of the diffusing matter. It follows that it must be possible, once u is given at any instant of time, to compute the density distribution at any later (and of course, also earlier) instant of time. In contrast to problems of ordinary hydrodynamics, the diffusion current in the system under consideration is thus completely determined by the assignment at an arbitrary instant of time of the density as a function of the coordinates, without simultaneously requiring the knowledge of current velocity as a function of the coordinates. This is due to the fact that the current velocity defined by Eq. (32) is a function of u and the coordinates alone, and does not depend on the history of the system. Thus, if $u(x, y, z)$ is known, then it also specifies $v(x, y, z)$, and therefore the evolution of the system in the following time step is completely determined in the sense of classic hydrodynamics.

We also note that a time reversal operation, an exchange of t with $-t$ in Eq. (3) is not possible because D, the diffusion coefficient, is positive-definite owing to its molecular theoretical meaning. The diffusion process is therefore "irreversible". This is also evident from the fact that the velocity current for a given u is a function of the position only which means that the initial velocities are not reversible and are determined solely by the collisions with the surrounding molecules.

The situation is quite different in the quantum mechanical case. Since the particle motion is not disturbed here by collisions with molecules in the surrounding matter, the motion of the particle cluster is essentially determined by the initial positions and velocities of the particles. It is therefore clear that there cannot be a differential equation for the density function w in the same way as it occurs for classic diffusion. That, on the contrary, an equation of the form (4) holds, can be most easily seen from the point of view of wave mechanics. From this point

[21] J. B. Johnson, *Phys. Rev.* **29,** 367, 1927; **32,** 97, 1928; N. H. Williams, *ibidem* **40,** 121, 1932; L. S. Ornstein, H. C. Burger, J. Taylor and W. Clarkson, *Proc. Roy. Soc. London (A)* **115,** 391, 1927.

of view, the particle cluster forms a "wave packet", i.e., a superposition of harmonic partial waves of the form

$$\psi_k = \varphi_k e^{2\,\pi\,i\,E_k\,t/h}$$

whose number has the cardinality of the continuum for the boundary conditions considered here. Here, φ_k stands for the "*amplitude function*": a complex function of the position of the form

$$\varphi_k = a_k e^{i\,S_k}$$

that contains two real functions of the position, the amplitude a_k[22] and the phase S_k. The assignment of all the a_k's and S_k's as functions of the position fully specifies the φ in the wave packet under consideration at a given instant of time, as well as for every later (or earlier) instant, as a consequence of the differential Eq. (4). This is physically obvious, since the fate of each partial wave is determined by the specification of amplitude and phase at time zero and thus the fate of the wave packet created by interference from partial waves is also determined. One immediately understands that to describe the state of the wave field one needs two scalars or one complex function: the Schrödinger function.

Since the density of the cluster under consideration (now considered from the corpuscular point of view) is specified solely by $|\psi|$ according to Eq. (5), the assignment of ψ as a function of the position entails more detailed information than the distribution of the particles' *positions* at a certain instant of time. According to what said above, as the fate of the cluster is determined by ψ, it is evident that the assignment of ψ also contains information about the distribution of the *velocities* at a certain instant of time. If, conversely, the initial velocities are not known, then it is not possible to make predictions from the initial distribution alone about the motion of the particle cluster. In fact, there cannot be a differential equation for $|\psi|$. Nevertheless, only the density $w = |\psi|^2$ or, interpreted as a virtual entity, the probability density of the position is observable and not ψ itself. This paradoxical state can be immediately explained as a consequence of the uncertainty relations. If ψ was indeed observable, then, according to our discussion, the position and velocity distribution would be simultaneously assigned for our particle cluster, which is not possible!

The fact that the coefficient on the left hand side of Eq. (4) must be purely imaginary or the diffusion coefficient ε in (9) must be purely imaginary can be seen as follows: if, at an arbitrary instant of time, the phases S_k of all the partial waves are reversed by 180°, then every φ_k turns into φ_k^* and therefore ψ turns into ψ^*. At the same time, however, the reversal of all phases means either turning all wave processes in the opposite direction or the complete reversal of the motion of the wave packet. The exchange of ψ with its conjugate complex value ψ^* is nothing more than a time reversal, and the differential Eq. (4), which ψ satisfies, must therefore remain unchanged under simultaneous replacement of ψ with ψ^* and of t with $-t$. This is actually only possible, provided that the Hamilton operator H is time independent, if the coefficient of $\frac{\partial \psi}{\partial t}$ is purely imaginary. The occurrence of the imaginary diffusion coefficient simply means, as Schrödinger has already pointed out[23], the reversibility of the quantum mechanical "diffusion" in contrast to the classical one, a discrepancy that was already emphasized in Sects. 2 and 3 as a part of the discussion of the differences between Eqs. (13) and (29).

Prague, January 1933.

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Funding Information Open Access funding provided by University of Helsinki including Helsinki University Central Hospital.

[22] **TN**: the original text here and below reads A_k.

[23] Erwin Schrödinger, *loc. cit.*, ref 5.

A Translation notes

A.1 Acknowledgement to K. Löwner

It is intriguing to read that Fürth acknowledges K. Löwner for hints in the derivation of the quantum uncertainty relation. Karel Löwner, also known as Charles Loewner after emigration to the U.S., was a mathematician whose work on conformal mappings led to the discovery of what is now commonly known as the "Loewner differential equation". The stochastic extension of his work is the stochastic Loewner equation or Schramm-Loewner evolution (SLE) a family of Markov processes describing interfaces in two-dimensional critical systems. The study of SLE has attracted a lot of attention both in the physics and mathematics communities over the last two decades (see, e.g., [23,24]).

B Commentary

In the following, we recast the derivation of Fürth's uncertainty relations in the language of stochastic differential equations driven by Wiener processes. We then hint at the similarities and differences between these uncertainty relations and the universal inequalities concerning currents, their variance and entropy production in fluctuating classical nano-systems that go under the name of the thermodynamic uncertainty relations (TUR). We also emphasize how I. Fényes [25] stressed the generality of the relations obtained by Fürth, paving the way for Nelson's "stochastic mechanics" program and for the Parisi-Wu stochastic quantization method.

B.1 Continuity of Brownian motion

A qualitative understanding of Fürth's result based on the present-day theory of Brownian motion is as follows. Paths of a Brownian motion are Hölder continuous with exponent 1/2 with probability one. This means that they are nowhere differentiable. As a consequence, if one observes a Brownian particle with increasing resolution of the particle position then at the same time one registers a boundless increase of the derivatives of the trajectory.

B.2 General derivation of the uncertainty relation

Here, we derive the generalization due to Fényes [25] of Fürth's uncertainty relations. In our derivation we do not make any reference to Quantum Mechanics. We instead adhere to Fürth's point of view that uncertainty relations are an indicator of the randomness of a dynamical system. As such, uncertainty relations may generically apply to stochastic processes thus permitting to address the

question of the limit of measurability with a measurement device.

also in the classical context. In the derivation we use the notation common in the modern theory of stochastic processes [26,27]. For the reader's convenience we explicitly relate the modern notation with Fürth's. We refer for a more complete mathematical discussion of the generic emergence of uncertainty relations in classical stochastic to [28] and to [29,30] for previous treatments more closely motivated by Nelson's stochastic mechanics [31].

We start by recalling the diffusion pathwise probabilistic definition of the current velocity and its relation with the "coefficients" of the Fokker-Planck equation.

Let us consider a stochastic process $\{\boldsymbol{\xi}_t\}_{t\geqq 0}$ with drift

$$\boldsymbol{b}(\boldsymbol{x},t)=\lim_{s\searrow 0} E\left(\frac{\boldsymbol{\xi}_{t+s}-\boldsymbol{\xi}_t}{s}\middle|\boldsymbol{\xi}_t=\boldsymbol{x}\right), \tag{B.1}$$

and diffusion

$$\mathbb{D}(\boldsymbol{x},t)=\lim_{s\searrow 0} E\left(\frac{(\boldsymbol{\xi}_{t+s}-\boldsymbol{\xi}_t)\otimes(\boldsymbol{\xi}_{t+s}-\boldsymbol{\xi}_t)}{s}\middle|\boldsymbol{\xi}_t=\boldsymbol{x}\right). \tag{B.2}$$

In writing the definition of drift and diffusion we used the symbol E to denote the expectation value with respect to the probability measure of the process. In other words, if $f\colon\mathbb{R}^d\mapsto\mathbb{R}$ is any test function and $\{\boldsymbol{\xi}_t\}_{t\geqq 0}$ takes values on the full Euclidean space $\mathbb{R}^d$ then

$$Ef(\boldsymbol{\xi}_t)=\int_{\mathbb{R}^d}\mathrm{d}^d x\, f(\boldsymbol{x})\,\mathscr{p}(\boldsymbol{x},t)$$

or in Fürth's notation

$$\bar{f}=Ef(\boldsymbol{\xi}_t)$$

The expectation values (B.1), (B.2) are subjected to the condition that at time t the stochastic process takes the value $\boldsymbol{x}$.

In order to proceed, we assume that both the drift and diffusion enjoy regularity properties in $\mathbb{R}^d$ such that the transition probability density $\mathscr{T}$ satisfies the Kolmogorov's forward (Fokker-Planck)

$$\partial_t \mathscr{T}(\boldsymbol{x},t|\boldsymbol{y},s) + \partial_{\boldsymbol{x}} \cdot \boldsymbol{b}(\boldsymbol{x},t)\mathscr{T}(\boldsymbol{x},t|\boldsymbol{y},s) = \frac{1}{2}\operatorname{Tr}\partial_{\boldsymbol{x}} \otimes \partial_{\boldsymbol{x}}\mathbb{D}(\boldsymbol{x},t)\mathscr{T}(\boldsymbol{x},t|\boldsymbol{y},s), \tag{B.3}$$

and backward equations [26]

$$\partial_s \mathscr{T}(\boldsymbol{x},t|\boldsymbol{y},s) + \boldsymbol{b}(\boldsymbol{y},s)\cdot\partial_{\boldsymbol{y}}\mathscr{T}(\boldsymbol{x},t|\boldsymbol{y},s) + \frac{1}{2}\operatorname{Tr}\mathbb{D}(\boldsymbol{y},s)\partial_{\boldsymbol{y}} \otimes \partial_{\boldsymbol{y}}\mathscr{T}(\boldsymbol{x},t|\boldsymbol{y},s) = 0, \tag{B.4}$$

both subject to the time boundary condition

$$\lim_{t-s\searrow 0}\mathscr{T}(\boldsymbol{x},t|\boldsymbol{y},s) = \delta^d(\boldsymbol{x}-\boldsymbol{y}),$$

for any $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^d$, and $t \geqq s \geqq 0$. In such a case, the probability density of the process $\{\boldsymbol{\xi}_t\}_{t\geqq 0}$ evolving from any reasonable data $p_\iota(\boldsymbol{x})$ at time zero

$$p(\boldsymbol{x},t) = \int_{\mathbb{R}^d}\mathrm{d}^d\boldsymbol{y}\;\mathscr{T}(\boldsymbol{x},t|\boldsymbol{y},0)\,p_\iota(\boldsymbol{y}), \qquad \forall\, t \geqq 0. \tag{B.5}$$

satisfies the Fokker-Planck equation and enjoys the Markov property

$$p(\boldsymbol{x},t) = \int_{\mathbb{R}^d}\mathrm{d}^d\boldsymbol{y}\;\mathscr{T}(\boldsymbol{x},t|\boldsymbol{y},s)p(\boldsymbol{y},s), \qquad \forall\, t \geqq s \geqq 0.$$

Under these hypotheses, the current velocity is defined as the conditional expectation of the time-symmetric increment (see, e.g., [4])

$$\boldsymbol{v}(\boldsymbol{x},t) = \lim_{s\searrow 0}\mathrm{E}\left(\frac{\boldsymbol{\xi}_{t+s}-\boldsymbol{\xi}_{t-s}}{2\,s}\middle|\boldsymbol{\xi}_t = \boldsymbol{x}\right). \tag{B.6}$$

Current velocity is the modern name for what Fürth calls the diffusion velocity, i.e., the quantity defined by Eq. (32) of Fürth's paper.

The evaluation of the conditional expectation (B.6) yields

$$\boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{b}(\boldsymbol{x},t) - \frac{1}{2\,p(\boldsymbol{x},t)}\partial_{\boldsymbol{x}}\left(\mathbb{D}(\boldsymbol{x},t)p(\boldsymbol{x},t)\right), \tag{B.7}$$

where p is the probability density of $\{\boldsymbol{\xi}_t\}_{t\geqq 0}$. It must be emphasized that the current velocity depends upon the initial distribution and the transition probability density $\mathscr{T}$ of the stochastic process in consequence of (B.5). In this sense, the current velocity is an integral quantity of the stochastic process in stark contrast to the drift (B.1) which is a local quantity independent of initial data.

One can prove (B.7) using Kolmogorov's *time-reversal relation* between the probability and the forward $\mathscr{T}$ and backward $\mathscr{T}_R$ transition probability densities, that

$$\mathscr{T}_R(\boldsymbol{y},t|\boldsymbol{x},t+s)p(\boldsymbol{x},t+s) = \mathscr{T}(\boldsymbol{x},t+s|\boldsymbol{y},t)p(\boldsymbol{y},t).$$

Indeed, under our hypotheses the identity

$$\begin{aligned}\mathrm{E}(\boldsymbol{\xi}_{t-s}|\boldsymbol{\xi}_t = \boldsymbol{x}) &= \int_{\mathbb{R}^d}\mathrm{d}^d y\;\boldsymbol{y}\;\mathscr{T}_R(\boldsymbol{y},t-s|\boldsymbol{x},t)\\ &= \int_{\mathbb{R}^d}\mathrm{d}^d y\;\boldsymbol{y}\,\frac{\mathscr{T}(\boldsymbol{x},t|\boldsymbol{y},t-s)p(\boldsymbol{y},t-s)}{p(\boldsymbol{x},t)}\end{aligned}$$

holds true. We then obtain (B.7), observing that as a function of $\boldsymbol{y}$, the density obeys the Fokker-Planck equation, whereas the transition probability satisfies Kolmogorov's backward Eq. (B.4).

A remarkable consequence of (B.7) is that upon expressing the Fokker-Planck equation satisfied by a density $p(\boldsymbol{x},t)$ in terms of the corresponding current velocity yields a mass continuity equation

$$\partial_t p(\boldsymbol{x},t) + \partial_{\boldsymbol{x}}\cdot\boldsymbol{v}(\boldsymbol{x},t)p(\boldsymbol{x},t) = 0.$$

Let us now consider the product of the variances of the position and current velocity processes respectively of $\{\boldsymbol{\xi}_t\}_{t\geqq 0}$:

$$\operatorname{Var}\boldsymbol{\xi}_t = \int_{\mathbb{R}^d}\mathrm{d}^d\boldsymbol{x}\;p(\boldsymbol{x},t)\,\|\boldsymbol{x}-\mathrm{E}\,\boldsymbol{\xi}_t\|^2,$$

and

$$\operatorname{Var} \boldsymbol{v}(\boldsymbol{\xi}_t, t) = \int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{x} \; p(\boldsymbol{x}, t) \left\| \boldsymbol{v}(\boldsymbol{x}, t) - \operatorname{E} \boldsymbol{v}(\boldsymbol{\xi}_t, t) \right\|^2 .$$

We observe that in Fürth's notation

$$\operatorname{Var} \boldsymbol{\xi}_t \equiv (\Delta \boldsymbol{\xi}_t)^2$$

The Cauchy-Schwarz inequality immediately yields

$$(\operatorname{Var} \boldsymbol{\xi}_t) \operatorname{Var} \boldsymbol{v}(\boldsymbol{\xi}_t, t) \geqq \left| \int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{x} \; p(\boldsymbol{x}, t) \, (\boldsymbol{x} - \operatorname{E} \boldsymbol{\xi}_t) \cdot (\boldsymbol{v}(\boldsymbol{x}, t) - \operatorname{E} \boldsymbol{v}(\boldsymbol{\xi}_t, t)) \right|^2 . \tag{B.8}$$

We then use (supposing that the density vanishes sufficiently fast at infinity)

$$\operatorname{E} \boldsymbol{v}(\boldsymbol{\xi}_t, t) = \operatorname{E} \boldsymbol{b}(\boldsymbol{\xi}_t, t)$$

to write

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{x} \; p(\boldsymbol{x}, t) \, (\boldsymbol{x} - \operatorname{E} \boldsymbol{\xi}_t) \cdot (\boldsymbol{v}(\boldsymbol{x}, t) - \operatorname{E} \boldsymbol{v}(\boldsymbol{\xi}_t, t)) \\ &\quad = \int_{\mathbb{R}^d} \mathrm{d}^d \boldsymbol{x} \; p(\boldsymbol{x}, t) \, (\boldsymbol{x} - \operatorname{E} \boldsymbol{\xi}_t) \cdot \left(\boldsymbol{b}(\boldsymbol{x}, t) - \operatorname{E} \boldsymbol{b}(\boldsymbol{\xi}_t, t) - \frac{1}{2 \, p(\boldsymbol{x}, t)} \partial_x \mathbb{D}(\boldsymbol{x}, t) p(\boldsymbol{x}, t) \right) . \end{aligned}$$

We thus arrive at the inequality:

$$(\operatorname{Var} \boldsymbol{\xi}_t) \; (\operatorname{Var} \boldsymbol{v}(\boldsymbol{\xi}_t, t)) \geqq \left| \operatorname{E} \left(\boldsymbol{\xi}_t \cdot \boldsymbol{b}(\boldsymbol{\xi}_t, t) \right) \cdot \operatorname{E} \boldsymbol{b}(\boldsymbol{\xi}_t, t) - (\operatorname{E} \boldsymbol{\xi}_t) + \frac{1}{2} \operatorname{E} \operatorname{Tr} \mathbb{D}(\boldsymbol{\xi}_t, t) \right|^2 . \tag{B.9}$$

This inequality is due to Fényes [25]. Fürth's results are recovered when the drift is negligible, i.e., when $\boldsymbol{b} = 0$, the relation reduces to

$$(\operatorname{Var} \boldsymbol{\xi}_t) \; (\operatorname{Var} \boldsymbol{v}(\boldsymbol{\xi}_t, t)) \geqq \frac{1}{4} \left| \operatorname{E} \operatorname{Tr} \mathbb{D}(\boldsymbol{\xi}_t, t) \right|^2 . \tag{B.10}$$

From the probabilistic point of view, the vanishing of the drift means that the stochastic process $\{\boldsymbol{\xi}_t\}_{t \geqq 0}$ is a local martingale of the Wiener process [27]. Additional regularity assumptions yield the martingale property, i.e., the fact that the expected value of the process obeys a statistical conservation law. In fact, it is possible to arrive at an inequality of the form (B.10) even for a general stochastic process with non-vanishing drift [28]. In this latter case, the inequality holds for any observable of the process that enjoys the mean-value martingale property and is therefore conserved on average.

On the other hand, at equilibrium $\boldsymbol{v} = 0$ by definition. Therefore, we can only expect that, in general,

$$(\operatorname{Var} \boldsymbol{\xi}_t) \; (\operatorname{Var} \boldsymbol{v}(\boldsymbol{\xi}_t, t)) \geqq 0.$$

To substantiate the last observation let us consider the case

$$\boldsymbol{b}(\boldsymbol{x}) = -\partial_x U(\boldsymbol{x}) \qquad \& \qquad \mathbb{D} = D \, \mathbb{1}.$$

Then equilibrium means (assuming U positive definite and confining)

$$p(\boldsymbol{x}) \propto e^{-\frac{2 \, U(\boldsymbol{x})}{D}},$$

so that

$$\begin{aligned} \int_{\mathbb{R}^d} \mathrm{d}^d x \; p(\boldsymbol{x}) \, \boldsymbol{x} \cdot \boldsymbol{b}(\boldsymbol{x}) &= - \int_{\mathbb{R}^d} \mathrm{d}^d x \; p(\boldsymbol{x}) \, \boldsymbol{x} \cdot \partial_x U(\boldsymbol{x}) \\ &= \frac{D}{2} \int_{\mathbb{R}^d} \mathrm{d}^d x \, \boldsymbol{x} \cdot \partial_x p(\boldsymbol{x}) = -\frac{D}{2} d, \end{aligned}$$

whereas

$$\int_{\mathbb{R}^d} \mathrm{d}^d x \; p(\boldsymbol{x}) \, \boldsymbol{b}(\boldsymbol{x}) = \frac{D}{2} \int_{\mathbb{R}^d} \mathrm{d}^d x \; \partial_x p(\boldsymbol{x}) = 0.$$

Thus the right-hand side of (B.9) vanishes.

B.3 Influence on stochastic mechanics

Fürth's paper is discussed in Fényes [25]. This paper lays down the foundation of what will be Nelson's "stochastic mechanics" program [4].

In the introduction of [25], Fényes states:

Although F ü r t h has demonstrated the existence in diffusion theory of a relation that is formally analogous to H e i s e n b e r g's, in his opinion the two relations cannot have the same meaning, because the F o k k e r equation cannot be valid in quantum mechanics.

Fényes' findings in [25] as summarized in the paper's abstract are:

There are also certain uncertainty relations for M a r k o v processes. A certain probability-amplitude function can also be assigned to a M a r k o v processes. The F o k k e r equation is also valid in quantum mechanics. The H e i s e n b e r g relation is a special case of the uncertainty relation of the M a r k o v processes. The wave-mechanical wave function is a special case of probability-amplitude functions governed by M a r k o v processes. The wave-mechanical processes are special M a r k o v processes. The H e i s e n b e r g relation is (in contrast to the previous interpretation) exclusively a consequence of the statistical approach, and is independent of the disturbances occurring in the two measurements.

Finally Fürth and Fényes are known in the stochastic quantization community where they are somewhat considered as precursors to the Parisi-Wu method (see, e.g., the discussion in the introduction of [6]).

C Comparison with the thermodynamic uncertainty relations

Over the last few years, universal inequalities establishing a lower bound for the variance of an empirical current $\mathscr{J}$ in terms of its square average and the net average entropy production S^{tot}

$$\operatorname{Var} \mathscr{J} \geqq \frac{2\,(\mathrm{E}\,\mathscr{J})^2}{e^{S^{\mathrm{tot}}/k_{\mathrm{B}}} - 1}, \tag{C.1}$$

have attracted considerable attention (see, e.g., [3,32] and refs therein). In the form written above the inequality is often referred to as the "generalized thermodynamic uncertainty relation" to discriminate from a "specialized" version of the inequality. The latter is formally obtained by retaining only the leading term of the expansion of the denominator in powers of the net average entropy production on the right hand side.

The derivation of (C.1) surmises that the joint probability distribution of the entropy production and the current satisfies a detailed fluctuation relation. The derivation is therefore based on statistical considerations whose validity can be ascertained for certain discrete-state Markov processes. Remarkably, the specialized version of the inequality admits a first principle yet perturbative derivation for near equilibrium processes [32]. It is thus instructive to compare these results with the statistical physics implications of Fürth's uncertainty relations.

First of all, Fürth's uncertainty relations arise from modeling classical open systems by means of stochastic differential equations. From the point of view of first principle derivations, the use of stochastic differential equations is justified by the scaling limits applied in the weak environment system coupling regime [33]. Hence, the validity domain essentially overlaps with that of thermodynamic uncertainty relations at least as far as general state-of-the-art mathematically rigorous considerations are concerned.

Secondly, Fürth's uncertainty relations involve the variance of the process and that of its current velocity. In particular, the current velocity is a zero-average quantity identically vanishing at equilibrium. Thus, the thermodynamic inequality (C.1) does not carry any non-trivial information about current velocity fluctuations. Fürth's uncertainty relations are close analogues of the quantum uncertainty relations, and, as their quantum counterparts, ultimately derive from the Cauchy-Schwartz inequality applied to the expectations of the products of the process and its velocities. The thermodynamic uncertainty relations, on the other hand, stem from the Cramér-Rao equality applied to the process itself, which connect the entropy production to its average and variance [34].

A relation between two types of uncertainty relations, however, exists if we turn the attention to entropic indicators of diffusion process. The connection between fluctuations of the current velocity $\boldsymbol{v}(\boldsymbol{\xi}, t)$ and entropy production is obtained by observing (see, e.g., [3, eq. (3.94)]) that, when $\mathbb{D} = D\,\mathbb{1}$,

$$S^{\mathrm{tot}}(t_{\mathrm{i}}, t_{\mathrm{f}}) = \mathrm{E} \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} \mathrm{d}t \, \|\boldsymbol{v}(\boldsymbol{\xi}_t, t)\|^2,$$

is proportional to the average entropy produced by a Langevin-Smoluchowski process $\boldsymbol{\xi}_t$ in the interval $t \in [t_{\mathrm{i}}, t_{\mathrm{i}}]$. In the case of entropic indicators, the expressions for the variance of martingale and for the quantity yielding the lower bound in the Fürth's uncertainty relation are straightforwardly linked. A simple application of the Cauchy-Schwarz inequality turns the uncertainty relation into a lower bound for the variance, e.g., of the current velocity fluctuations transporting the probability distribution of the entropy production during a non-equilibrium transition. In addition, the lower bound can be expressed in terms of the average entropy production occurring during the transition. In this manner, quantities entering into Fürth's and the thermodynamic uncertainty relations become connected. The conclusion is that the formulation of Fürth's uncertainty relation in the modern language of the theory of stochastic processes offers an avenue to estimate, at least from below, non-equilibrium indicators that are otherwise difficult to access experimentally.

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