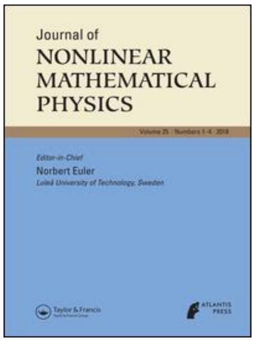

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Symmetry classification of scalar Ito equations with multiplicative noise

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We provide a symmetry classification of scalar stochastic equations with multiplicative noise. These equations can be integrated by means of the Kozlov procedure, by passing to symmetry adapted variables.

Keywords: Symmetry; Stochastic differential equations; Multiplicative noise.

2000 Mathematics Subject Classification: 34F05, 37H10, 58D19.

1. Introduction

In a recent paper [1] we have explicitly integrated the *stochastic logistic equation with multiplicative noise*

$$dx = \left(A x - B x^2\right) dt + \mu x dw , \tag{1.1}$$

where $A,\ B,\ \mu$ are positive real constants. The motivation for this was provided by questions in Mathematical Biology, in particular Population Dynamics, where the logistic equation plays a central role. In this context, multiplicative noise is also known as *environmental noise*, as it models fluctuations due to changes in the environmental conditions and thus acting in the same way – and in a fully correlated manner – on all the individuals [2, 3]. (Uncorrelated fluctuations for different individuals give raise to noise terms proportional to $\sqrt{x}$; in that context, one refers to this as *demographic noise* [2].)

In order to integrate (1.1), we have employed tools from the recently developed theory of *symmetry of stochastic differential equations* [4–18]. In particular, it was found that (1.1) admits a simple Lie symmetry and hence, thanks to a general constructive theorem by Kozlov [7–9] (see also [11–13, 17]), one can pass to symmetry-adapted variables allowing for a direct integration. See [1] for details.[a]

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[a]We assume the reader to be familiar with the standard theory of symmetry of (deterministic) differential equations [19–24] and with stochastic differential equations [25–31]; so we will not discuss these topics.

It is natural to wonder if the integrability of (1.1) was a lucky accident or, as it indeed appears more probable, (1.1) is one of a more and less wide family of integrable equations.

In the present work, we want to consider scalar Ito equations (we always assume the drift $f(x,t)$ and noise $\sigma(x,t)$ are smooth functions of their arguments)

$$dx = f(x,t)\,dt + \sigma(x,t)\,dw \tag{1.2}$$

with a noise term $\sigma(x,t)$ corresponding to multiplicative noise, i.e.

$$\sigma(x,t) = S(t)\,x\,;$$

we assume $S(t) \not\equiv 0$, or we would have a deterministic equation. For equations in this class, we want to classify those admitting a simple Lie-point symmetry and which hence can be integrated by means of the Kozlov approach.

Thus the general form of equations to be considered is

$$dx = f(x,t)\,dt + S(t)\,x\,dw\,. \tag{1.3}$$

We will look for the most general symmetries allowed by our classification [10], i.e. W-symmetries; we will however restrict our attention to *simple* W-symmetries, as these are the only ones which can be used to integrate the equation [18]. That is, we look for symmetry vector fields of the form

$$X = \varphi(x,t,w)\,\partial_x + R\,w\,\partial_w\,, \tag{1.4}$$

where R is a real constant.

In this case the determining equations read [18]

$$\varphi_t + f\,\varphi_x - \varphi\,f_x + \frac{1}{2}\triangle(\varphi) = 0, \tag{1.5}$$

$$\varphi_w + \sigma\,\varphi_x - \varphi\,\sigma_x = R\,\sigma\,. \tag{1.6}$$

Here and below $\triangle$ is the Ito Laplacian, which in our scalar case reads simply

$$\triangle(\Psi) := \frac{\partial^2\Psi}{\partial w^2} + 2\,\sigma\,\frac{\partial^2\Psi}{\partial x\partial w} + \sigma^2\,\frac{\partial^2\Psi}{\partial x^2}\,. \tag{1.7}$$

Remark 1. We recall that, as shown by Kozlov [7–9] (see also [10, 18] for the extension of Kozlov theorem to W-symmetries), once we have determined a simple symmetry of an Ito SDE, this is explicitly integrated by passing to the (symmetry adapted) new variable

$$y = \int \frac{1}{\varphi(x,t,w)}\,dx\,; \tag{1.8}$$

thus our classification of equations (with multiplicative noise) admitting a simple symmetry also provides a classification of equations (in this class) which can be explicitly integrated by the Kozlov change of variables.

Remark 2. The paper [8] also provides a symmetry classification of scalar SDEs; this refers to *standard deterministic* symmetries, while the present one deals with the (more general) framework of *W-symmetries*, including also the case of *standard random* symmetries [10]. On the other hand, in [8] one considers general scalar Ito equations, while we only deal with a specific form of the diffusion coefficient $\sigma(x,t)$.

2. The basic classification

The determining equations for our specific form of Ito equation with multiplicative noise (1.3) restrict the scenario – in terms of possible drifts term – for which symmetries are present: only when the drift $f(x,t)$ has a specific form, the deterministic equations allow solutions φ, i.e. symmetries (standard or W). At this point we do not want to discuss about regularity requirements of the functions involved, but we will rather derive the constrained form of $f(x,t)$. Thus, from now on, we will assume all functions we meet to be regular enough. In the following computation we will get the aforementioned constraints on $f(x,t)$, summarized in Proposition 1, by solving the determining equations by separation of variables.

For the specific form of (1.3), the second determining equation (1.6) reads

$$\varphi_w + S(x\,\varphi_x - \varphi - Rx) = 0\,. \tag{2.1}$$

The general solution of this is[b]

$$\varphi(x,t,w) = x\left[R\,\log(x) + q(t,z)\right]\,, \tag{2.2}$$

where we have written

$$z := w - \frac{\log(x)}{S(t)}\,. \tag{2.3}$$

Restricting (1.5) to functions of this type, i.e. plugging (2.2) into (1.5), we get an equation of the form

$$x\,q_t + \chi_0 + \chi_1\,q + \chi_2\,q_z = 0\,, \tag{2.4}$$

where we have written

$$\begin{aligned}
\chi_0 &= \frac{R}{2}\,\left[2\,f\,[1+\log(x)] + x\,\left(S^2 - 2\log(x)\,f_x\right)\right]\,,\\
\chi_1 &= (f - x\,f_x)\,,\\
\chi_2 &= \frac{1}{2}\,x\,S + \frac{x\,S'}{S^2}\,\log(x) - \frac{f}{S}\,.
\end{aligned}$$

Here both the functions $f(x,t)$ and $q(t,z)$ are unknown. Moreover, now our set of variables is (x,t,z).

Differentiating (2.4) three times in x and once in z, eliminating a common factor $(x^2 S(t))^{-1}$ (as stressed above $S(t)\neq 0$), and writing

$$g(x,t) := f_{xxx}(x,t) \tag{2.5}$$

for ease of notation, we get

$$q_{zz}(t,z)\,\left[S'(t) + x^2\,S(t)\,g(x,t)\right] + q_z(t,z)\,\left[2x^2\,S^2(t)\,g(x,t) + x^3\,S^2(t)\,g_x(x,t)\right] = 0\,. \tag{2.6}$$

Now we can separate the x and z variables, which yields

$$-\frac{q_{zz}}{q_z} = \frac{2x^2\,S^2(t)\,g(x,t) + x^3\,S^2(t)\,g_x(x,t)}{S'(t) + x^2\,S(t)\,g(x,t)}\,. \tag{2.7}$$

[b]Note that by $\log(x)$ we will always mean the natural logarithm; moreover it will be understood that we write $\log(x)$ for $\log(|x|)$. This is specially justified when we look at equations like (1.1), with x describing a population – which is by definition non-negative – which is the motivation of our work.

Here the l.h.s. is a function of z and t, while the r.h.s. is a function of x and t. Thus this relation can hold only if both sides are a function of t alone, i.e. if the l.h.s. is independent of z and the r.h.s. is independent of x.

Requiring that the l.h.s. of (2.7) is independent of z amounts to solving

$$q_{zz}^2 = q_z\, q_{zzz}\,; \tag{2.8}$$

this yields immediately (here $Q_i(t)$ are arbitrary smooth functions)

$$q(t,z) = \exp[z\, Q_1(t)]\, Q_2(t) + Q_3(t)\,. \tag{2.9}$$

Remark 3. Note that we can exclude the case $Q_1(t) = 0$. In fact, if this was the case we would have $q(t,z) = Q_2(t) + Q_3(t)$, but as these are both arbitrary functions, it is the same as assuming $q(t,z) = Q_3(t)$, which is obtained for $Q_2(t) = 0$.

As for the request that the r.h.s. of (2.7) is independent of x, this leads to

$$4\,S'\,g + x^3\,S\,g\,g_x + 5\,x\,S'\,g_x - x^4\,S\,g_x^2 + x^4\,S\,g\,g_{xx} + x^2\,S'\,g_{xx} = 0\,. \tag{2.10}$$

This equation can be solved, yielding the general solution

$$g(x,t) = \frac{1}{x^2\,S(t)}\,\left[x^{\alpha(t)}\,\eta(t) - S'(t)\right]\,, \tag{2.11}$$

where again α and η are arbitrary smooth functions.

Remark 4. We pause a moment to note that in order to determine $f(x,t)$ from a $g(x,t)$ of this general form, we should solve (2.5) as an equation for f. In doing this, we should pay attention to the arbitrary function $\alpha(t)$: it turns out that if $\alpha(t)$ is actually constant and takes either of the values $\{-1,0,+1\}$ we have special cases (one of the three integrations in x concerns a factor $1/x$ and hence produces a logarithm), as discussed below.

We can now go back to (2.7); with the expression obtained above for $q(t,z)$ and $g(x,t)$, this reads

$$Q_1(t) = -\alpha(t)\,S(t)\,. \tag{2.12}$$

Thus we set $Q_1(t)$ to be as in (2.12), and with this we get

$$q(t,z) = \exp\left[-\alpha(t)\,S(t)\,z\right]\,Q_2(t) + Q_3(t)\,. \tag{2.13}$$

Remark 5. It may be noted that in this way we constrain the expression for (the possible) φ; in particular, we have

$$\varphi(x,t) = x\,\left[R\,\log(x) + x^{\alpha(t)}\,\exp[-\alpha(t)S(t)w]\,Q_2(t) + Q_3(t)\right]\,. \tag{2.14}$$

Thus φ can indeed depend on w (unless $\alpha(t)$ or $Q_2(t)$ are zero). Note also that (at least at this stage) we can have proper W-symmetries, i.e. $R \neq 0$; moreover the possible dependence on $\log(x)$ is related to $R \neq 0$ – thus standard symmetries will not have any dependence on $\log(x)$.

Let us now go back to considering $f(x,t)$, i.e. the possible form of the symmetric Ito equations. Now that we have an expression for $g(x,t)$ the form of $f(x,t)$ is obtained by integrating it, i.e. solving (2.5) as an equation for f.

For the generic case, which in this context means $\alpha(t) \neq -1, 0, +1$ (these cases should be dealt with separately), this yields

$$f(x,t) = \left[F_1(t) + x F_2(t) + x^2 F_3(t)\right] + \Psi(x,t) \tag{2.15}$$

where we have defined

$$\Psi(x,t) := \left[(\log(x) - 1\, x \frac{S'(t)}{S(t)} - \Phi(x,t)\, \frac{\eta(t)}{S(t)}\right], \tag{2.16}$$

$$\Phi(x,t) := \frac{x^{1+\alpha(t)}}{[1-\alpha(t)]\,\alpha(t)\,[1+\alpha(t)]}\,. \tag{2.17}$$

As mentioned above, this expression holds provided $\alpha(t)$ is not constant and equal to either 0 or ± 1. In these special cases, the expression (2.15) still holds[c], but now $\Phi(x,t)$ should be defined in a different way:

$$\alpha(t) = -1 \quad \Phi(x,t) \to \Phi_{(-)}(x,t) := -\frac{1}{2}\log(x)\,, \tag{2.18}$$

$$\alpha(t) = \ 0 \quad \Phi(x,t) \to \Phi_{(0)}(x,t) := -x\,[1-\log(x)]\,, \tag{2.19}$$

$$\alpha(t) = +1 \quad \Phi(x,t) \to \Phi_{(+)}(x,t) := -\frac{1}{4}x^2\,[2\log(x) - 3]\,. \tag{2.20}$$

We can insert these expressions in that for $f(x,t)$; one should recall here that the $F_k(t)$ are generic functions, so that adding to them any other function of time will not change their nature but just change them from $F_k(t)$ to some other function $\widehat{F_k}(t)$, which we will still write as $F_k(t)$.

In this way, and writing for ease of notation

$$\Sigma(t) := \frac{S'(t)}{S(t)}\,, \quad \mathscr{F}(x,t) = F_1(t) + x F_2(t) + x^2 F_3(t)\,,$$

we have

$$f(x,t) = \begin{cases} \mathscr{F}(x,t) + G(t)\, x^{1+\alpha(t)} + \Sigma(t)\, x \log x & (\alpha(t) \text{generic}) \\ \mathscr{F}(x,t) + \Sigma(t)\, x \log x + H(t)\, x^2 \log x & (\alpha(t) = +1) \\ \mathscr{F}(x,t) + H(t)\, x \log x & (\alpha(t) = 0) \\ \left[F_1(t) + x\, F_2(t) \ + \ x^2\, F_3(t)\right] + H(t) \log x & (\alpha(t) = -1) \end{cases} \tag{2.21}$$

Here $G(t)$ and $H(t)$ are arbitrary functions.

Before going on to discuss the different cases, we note that our discussion so far already identifies the – rather restricted – general class of equations of the form (1.3) which *could* admit some symmetry. We summarize the result of our discussion in this respect in the following Proposition 1; note that in there we have used the presence of arbitrary functions, and also write

$$\mathscr{H}(x,t) = H_1(t) + x H_2(t) + x^2 H_3(t)\,,$$

in order to further simplify the writing of our functional forms.

[c] Indeed the term in square brackets in the r.h.s. of (2.15) correspond to the constants of integration – which of course depend on t – in the three x integrations.

Proposition 1. *Scalar SDEs of the form* (1.3) *may admit a symmetry only if* $f(x,t)$ *is of the general form*

$$f(x,t) = \mathscr{F}(x,t) + G(t)\,x^{1+\alpha(t)} + \mathscr{H}(x,t)\,\log x\,. \tag{2.22}$$

Remark 6. It may be worth mentioning that this is a general form and provides necessary, but by no means sufficient, conditions for the existence of a symmetry; under closer scrutiny we will find out that actually this is too general to provide also sufficient conditions: e.g., a symmetry will be present only if $H_3(t) = 0$.

Let us come back to our task of classifying symmetric equations and symmetries. We should now insert these expressions for $q(t,z)$ and $f(x,t)$ in (1.5). We obtain quite complicate expressions – which we do not report here – in the generic and in the three special cases.

The relevant point is that all dependencies on x – including those in $\log(x)$ – and z are now explicit, so that coefficients of different monomials in $\{z, x, \log(x)\}$ should vanish separately, and the equation splits in several simpler ones.

Remark 7. It should also be noted that in our equations we have some factors $x^{\alpha(t)}$, see (2.11); if $\alpha(t)$ is actually a constant and in particular if $\alpha(t)$ is an integer, this can interact with other terms. This can actually happen only for $\alpha(t) = 0, \pm 1$, see (2.22), and we have already warned that these should be treated separately (exactly for this reason); we will thus refer to the case where $\alpha(t) \neq 0, \pm 1$ as the *generic* case, and to those where $\alpha(t) = 0, \pm 1$ as *special* cases.

Remark 8. Note that in the special cases we can always set $\eta(t) = 0$ with no loss of generality.

3. Results

We will give here our results; the proof, which consists largely of a detailed computation, is confined to the "supplementary material" available online [32].

In order to state our results more compactly, it will be convenient to introduce the notations

$$\Gamma_{\pm}(t) := \frac{S^2(t)}{2} \pm \frac{S(t)}{k}\,[\Sigma(t) - \chi(t)]\,; \quad \chi(t) := \frac{Q'(t)}{Q(t)}\,. \tag{3.1}$$

Theorem 1. *The equations of the form* (1.3) *admit standard* ($R = 0$) *or W-symmetries* ($R \neq 0$) *for* $f(x,t)$ *given by one of the expressions in the following table, where* $\mathscr{F}$ *and* $\mathscr{G}$ *and* θ *are arbitrary functions* (*possibly zero*):

case	$S(t)$	R	$f(x,t)$
(a)	$S(t)$	0	$\Gamma_+(t)\,x + \Sigma(t)\,x\,\log x + \mathscr{G}(t)\,x^{1+k/S(t)}$
(b)	$S(t)$	0	$\Gamma_+(t)\,x + \Sigma(t)\,x\,\log x$
(c)	$S(t)$	0	$\mathscr{F}(t)\,x + \Sigma(t)\,x\,\log x$
(d)	$S(t)$	0	$\mathscr{F}(t)\,x + \theta(t)\,x\,\log x$
(e)	$S(t)$	R	$-\Gamma_-(t)\,x + \Sigma(t)\,x\,\log x$
(f)	s_0	0	$\mathscr{F}(t)\,x^2 + [s_0^2/2 - \chi(t)]\,x$
(g)	s_0	0	$\mathscr{F}(t) + [s_0^2/2 - \chi(t)]\,x$
(h)	s_0	R	$[s_0^2/2 - \chi(t)]\,x$

Their (*standard or W*) *symmetries are given by* (1.4) *with R as in the previous table and coefficient φ given by the following table, where $k \neq 0$ and K* (*possibly $K = 0$*) *are arbitrary constants:*

case	R	$\varphi(x,t,w)$
(a)	0	$Q(t)\,e^{-kw}\,x^{1+k/S(t)}$
(b)	0	$K\,S(t) + Q(t)\,e^{-kw}\,x^{1+k/S(t)}$
(c)	0	$K\,S(t)\,x$
(d)	0	$\theta(t)\,x$
(e)	R	$R\left[\mathscr{F}(t)\,e^{-kw}x^{1+k/S(t)} + \theta(t)\,x + x\log x\right]$
(f)	0	$K\,s_0\,x + Q(t)\,e^{-s_0 w}\,x^2$
(g)	0	$K\,x + Q(t)\,e^{s_0 w}$
(h)	R	$R[\theta(t)x + J(t)e^{s_0 w}x^2 + x\log x]$

Furthermore, no other scalar Ito stochastic differential equations (1.3) *beside those detailed here admit standard or W-symmetries.*

Proof. As mentioned before, the proof consists in explicitly computing and carrying out, starting from Proposition 1 and the determining equations, the possible symmetries listed in the table above. Checking that these equations admit the given symmetries just requires direct elementary computations, i.e. checking that the determining equations (1.5), (1.6) are satisfied. The second part of the statement, i.e. that these are the only equations admitting symmetries, requires more involved computations. For a sake of readability, these computations are given explicitly in the supplementary material. □

4. Discussion and Conclusions

We have considered in full generality equations of the form

$$dx = f(x,t)\,dt + S(t)\,x\,dw\,,$$

i.e. scalar stochastic differential equations with multiplicative noise, also known in population dynamics as equations with environmental noise.

We have investigated in which cases, i.e. for which functions $f(x,t)$ and $S(t)$, they admit a symmetry (or more precisely a one-parameter group of symmetries), setting this search within the most general class of symmetries, i.e. W-symmetries.

Our detailed analysis had to consider a generic case and three special cases, depending on the function $\alpha(t)$ introduced in (2.11); the special cases correspond to $\alpha(t)$ being a constant function with value $\{-1, 0, +1\}$.

We have found that the most general functional form of $f(x,t)$ admitting a symmetry (including the special cases) is

$$f(x,t) = \mathscr{F}(t)\,x + \mathscr{G}(t)\,x^{1+\alpha(t)} + \mathscr{H}(t)\,x\,\log(x)\,. \tag{4.1}$$

As for the symmetries, these are characterized by R and ϕ, see (1.4); the most general form of φ turned out to be

$$\varphi(x,t,w) = \mathscr{A}(t)\,x + \mathscr{B}(t)\,\exp[k\,w]\,x^{1+\alpha(t)} + R\,x\,\log x\,. \tag{4.2}$$

The reader is referred to Theorem 1 for the specific form taken by the functions $\{\mathscr{F}(t), \mathscr{G}(t), \mathscr{H}(t); \mathscr{A}(t), \mathscr{B}(t)\}$ as well as for $\alpha(t)$ and $S(t)$, which are in many cases constrained; in particular, several of the specific cases obtained above require that $S(t)$ is constant. Note also that in all subcases admitting such terms but one, we had $\mathscr{H}(t) = [S'(t)/S(t)]$; this also means that the terms $x \log x$ can be allowed in $f(x,t)$ only if $S(t)$ is actually constant.

It should be stressed that our work points out at a substantial difference with respect to the deterministic case: in fact, in that case the functional form of deterministic equations admitting symmetries (and of the symmetries) is largely unconstrained, and actually for the closet analogue to the equations considered here, i.e. for dynamical systems, it is known that we always have symmetries (albeit these can not be determined algorithmically). We have found that the situation is completely different for stochastic equations: not only symmetry is a non-generic property, but the functional form of stochastic equations admitting symmetries is severely constrained. This is maybe the most relevant – and qualitative – result of our classification; unfortunately it also means that, at least for scalar equations, the value of symmetry methods is restrained to a rather specific class of equations.

Our classification would naturally call for further work:

- It follows from the general results of Kozlov theory [7–9] (in some cases, extended to include W-symmetries as well) that equations of the form (4.1) – in some cases with extra conditions on $S(t)$ – can be integrated by passing to symmetry adapted variables. Thus each of our symmetric equations listed in Theorem 1 could be integrated by using the corresponding symmetries.
- An alternative – albeit strongly related – approach to integration of stochastic equations is that based on stochastic invariants [14–16]; it would be interesting to have a similar classification for invariants of scalar stochastic equations with environmental noise.
- We have classified equations with *environmental* noise; it would be natural to consider the same task for equations with *demographic* noise, or equations with both environmental and demographic noise (the so called *complete models* [2]).

These aspects lay outside our present scope and were not discussed here; we hope to tackle them in future work.

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