

## ON THE GEOMETRY OF TWISTED PROLONGATIONS, AND DYNAMICAL SYSTEMS

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ABSTRACT. I give a short review of the theory of twisted symmetries of differential equations, emphasizing geometrical aspects. Some open problems are also mentioned.

*Dedicated to Juergen Scheurle on the occasion of his retirement*

**1. Introduction.** Sophus Lie created what is nowadays known as the theory of Lie groups and algebras first and foremost to study (nonlinear) differential equations. Its theory has then been extended in several directions, in particular generalizing the set of admitted vector fields. On the other hand, it remained clear that once we have defined how the basic (independent and dependent) variables are acted upon by our transformations, the action on derivatives is given – by a natural action, known in geometrical terms as the *prolongation* operation.

More recently, it has been realized that one can also deform the action on derivatives, i.e. deform the prolongation operation (in this case one usually speaks of “twisted prolongation” and “twisted symmetry”), and still obtain useful concepts and results – where useful is meant in the sense of “useful to get solutions of the equations under study”, beside the abstract geometrical interest.

It happens that in these cases what is the deformation is assigned at the level of first derivatives, while deformations on higher derivative sees no new actor. This means that – as is often the case in symmetry theory of differential equations – in the case of first order equations, even more so for Dynamical Systems, one has “too much freedom” (a standard euphemism to mean there is no algorithmic way to proceed). Despite this fact, the theory can also be used in the context of dynamical systems (a special attention in this direction was paid in the development of  $\sigma$ -symmetries, see below).

In this paper I will review the theory of twisted symmetries, paying special attention to geometrical aspects – in particular to the connection between the usual Lie reduction and Lie-Frobenius one – and to results which can be applied in the realm of ODEs and Dynamical Systems. The Bibliography will provide the interest reader with indications on how to go beyond these short notes.

**2. Standard symmetries of differential equations.** I will consider differential equations<sup>1</sup> with independent variables  $x^i$  ( $i = 1, \dots, n$ ) and dependent variables  $u^a$

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<sup>1</sup>For the moment, ODEs or PDEs will not make a difference, and differential equations (DEs), are always possibly vector ones, i.e. systems; similarly, functions are always possibly vector ones – albeit in some cases I will use vector indices explicitly to avoid possible confusion.

( $a = 1, \dots, m$ ); partial derivatives will be denoted by  $u_J^a$ , where  $J$  is a multi-index  $J = \{j_1, \dots, j_n\}$  of order  $|J| = j_1, \dots, j_n$  and

$$u_J^a = \frac{\partial^{|J|} u^a}{\partial x_1^{j_1} \dots \partial x_n^{j_n}} \quad (1)$$

(here and somewhere in the following we moved the vector index of the  $x$  for typographical convenience). We denote by  $u_{(k)}$  the set of all partial derivatives of order  $k$ , and by  $u_{[n]}$  the set of all partial derivatives of order  $k \leq n$ .

**2.1. Geometrical description of differential equations and solutions.** The  $x$  are local coordinates in a manifold  $B$ , while  $u$  are local coordinates in a manifold  $U$ ; we consider the phase manifold  $M = B \times U$ , which has a natural structure of bundle  $(M, \pi_0, B)$  over  $B$  with fiber  $U$ .

As well known [1, 13, 37, 57, 58, 70] we can associate to  $(M, \pi_0, B)$  its Jet bundles  $J^k M$  of any order; these have a structure of fiber bundle  $(J^k M, \pi_k, B)$  over  $B$  but also of fiber bundle over those of lower order,  $(J^k M, \chi_{kq}, J^q M)$  with  $\pi_k = \pi_q \circ \chi_{kq}$ . Natural local coordinates in  $J^k M$  are provided by  $(x, u, u_{(1)}, \dots, u_{(k)})$ .

We also recall that the Jet bundle is equipped with a *contact structure*  $\Omega$  [4, 33, 58, 68, 71]; this can be encoded in the *contact forms*

$$\omega_J^a := du_J^a - u_{J,i}^a dx^i. \quad (2)$$

Functions  $u = f(x)$  are naturally identified with sections in  $M$  (elements of  $\Sigma(M)$ ); the function  $u = f(x)$  corresponds to the section

$$\gamma_f = \{(x, u) \in M : u = f(x)\}. \quad (3)$$

Note that in this way we have established a correspondence between an *analytical* object (the function) and a *geometrical* one (the section).

A section  $\gamma_f \in \Sigma(M)$  identifies naturally a section  $\gamma_f^{(k)}$  in  $\Sigma(J^k M)$ , with of course

$$\gamma_f^{(k)} = \{(x, u_{[k]}) \in J^k M : u_J^a = f_J^a(x) \quad \forall J : |J| \leq k\}. \quad (4)$$

Given a DE  $\Delta$  of order  $k$ , written in local coordinates as

$$F^i(x, u, \dots, u_{(k)}) = 0, \quad (5)$$

we consider its *solution manifold*  $S_\Delta \subset J^k M$ ,

$$S_\Delta = \{(x, u_{(1)}, \dots, u_{(k)}) : F(x, u_{[k]}) = 0\}. \quad (6)$$

This is just the set of points in  $J^k M$  where the relation described by  $\Delta$  between independent, dependent variables and derivatives is satisfied; but now we have again transformed an *analytical* object (the differential equation) into a *geometric* one.

The same can be done for the concept of solutions to  $\Delta$ : a function  $u = f(x)$  identifies, as mentioned above, a section  $\gamma_f$  in  $(M, \pi_0, B)$ , and this in turn identifies a section  $\gamma_f^{(1)}$  in  $(J^k M, \pi_k, B)$ , which is just the set of points  $(x, u_{[k]})$  with  $u^a = f^a(x)$  and  $u_J^a = f_J^a(x)$ . Now  $u = f(x)$  is a solution to  $\Delta$  if and only if  $\gamma_f^{(k)} \subset S_\Delta$ .

**2.2. Vector fields and prolongations.** Let us now consider a vector field  $X$  in  $M$  and its *prolongation* to  $J^k M$ . In local coordinates, we write

$$X = \sum_i \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_a \varphi^a(x, u) \frac{\partial}{\partial u^a} = \xi^i \partial_i + \varphi^a \partial_a ; \quad (7)$$

the prolongation is then described –in the same local coordinates – by

$$X^{(k)} = X + \sum_a \sum_{|J|=1}^k \psi_J^a \frac{\partial}{\partial u_J^a} := X + \psi_J^a \partial_a^J , \quad (8)$$

where we introduced the shorthand notations

$$\partial_i := \frac{\partial}{\partial x^i} , \quad \partial_a := \frac{\partial}{\partial u^a} ; \quad \partial_a^J := \frac{\partial}{\partial u_J^a} . \quad (9)$$

We will also write  $\psi_0^a = \varphi^a$ .

The coefficient  $\psi_J^a$  of the Jet components, i.e. of the components making the prolongation of  $X$ , are computed by the *prolongation formula*, which is more conveniently expressed in recursive form:

$$\psi_{J,i}^a = D_i \psi_J^a - u_{J,k}^a D_i \xi^k . \quad (10)$$

Here and in the following  $\tilde{J} = \{J, i\}$  is the multi-index with components  $\tilde{j}_m = j_m + \delta_{m,i}$ .

It is maybe worth recalling that this is easily obtained in analytic terms (we assume the reader to be familiar with this derivation, which is however provided in [21, 57, 58]), but the prolongation of a vector field can also be defined geometrically.

The following Lemmas are well known; see e.g. [21, 57, 58] for proofs and details.

**Lemma 1.** *The prolonged vector field  $X^{(n)}$  is the unique vector field in  $J^n M$  which: (i) is projectable to each  $J^k M$  for  $0 \leq k \leq n$ ; (ii) coincides with  $X$  when restricted to  $M$ ; (iii) preserves the contact structure on  $J^n M$ .*

**Lemma 2.** *The prolongation of the commutator of two vector fields is the commutator of their prolongations; in other words,*

$$[X, Y] = Z \Leftrightarrow [X^{(n)}, Y^{(n)}] = Z^{(n)} . \quad (11)$$

**Lemma 3.** *Let  $\eta : M \rightarrow \mathbf{R}$  be a differential invariant of order zero and  $\zeta : J^k M \rightarrow \mathbf{R}$  a differential invariant of order  $k$  for  $X^{(n)}$  ( $n > k$ ). Then  $\chi = (D_i \zeta / D_i \eta) : J^{k+1} M \rightarrow \mathbf{R}$  is a differential invariant of order  $k+1$  for  $X^{(n)}$ .*

**Remark 1.** Lemma 2 is also formulated saying that prolongation preserves Lie algebra structures.

**Remark 2.** The property stated in Lemma 3 is also known as “invariant by differentiation property” (IBDP); if we start with a complete  $\zeta$  of order 1, we can generate differential invariants (DIs) of all orders. In the case of ODEs, if we start with a complete set of DIs of order zero and one, we can generate in this way a complete set of DIs of any order, basically because if  $U$  is  $q$ -dimensional, we have  $k \cdot q$  DIs of order  $k$  (these includes those of lower orders), as follows at once from  $J^k M$  being of dimension  $d_k = (k+1)q + 1$ . The situation is different for PDEs, as the dimension of  $J^k M$  grows combinatorially; see e.g. the discussion in [57].

**2.3. Lie-point symmetries.** A (Lie-point<sup>2</sup>) symmetry, or more precisely a Lie-point symmetry generator, is a vector field  $X$  on  $M$  such that its *prolongation*  $X^{(k)}$  to  $J^k M$  is tangent to  $S_\Delta$ . For a given  $\Delta$  in the form (5), this condition is expressed in local coordinates as

$$\left[ X^{(k)} (F^i) \right]_{S_\Delta} = 0. \quad (12)$$

In these equations, also called *determining equations*, the  $F$  are given and one looks for  $\xi, \varphi$  (i.e. components of the vector field  $X$ ) satisfying them. As the components  $\psi_J^a$  of  $X^{(k)}$  along  $u_J$  are given in terms of  $\xi, \varphi$  and their derivatives, all dependencies of  $u_J^a$  (with  $|J| \neq 0$ ) are fully explicit, and hence (12) decouple into a system of simpler equations, one for each monomial in the  $u_J^a$ ; each of these is a *linear* PDE for the  $\xi$  and  $\varphi$ , and they can be solved algorithmically – usually with the help of a symbolic manipulation program, as the dimension of the system can be quite large.

**Remark 3.** The symmetry relation is a tangent one; this means that only *integral curves* of vector fields are relevant, and not the speed on these [17, 19, 22, 63].

**Remark 4.** A generic equation will have *no* symmetries; symmetry is a non-generic property – albeit it may become generic in a given class of equations: e.g., equations for isolated physical systems are invariant under time and space translations, and space rotations; as well know, conservation of Energy, Momentum and Angular Momentum is related to these invariances via Noether theorem [2, 36, 57].

**Remark 5.** There can be vector fields  $X$  such that (12) is satisfied without the restriction to  $S_\Delta$ , i.e. such that  $X^{(k)}(F) = 0$ ; in this case one speaks of *strong* (Lie-point) symmetries. The relation between strong and standard symmetries was clarified by Carinena, Del Olmo and Winternitz [7] (CDW theorem); roughly speaking – and up to some cohomological considerations – if a DE  $\Delta$  admits  $X$  as a symmetry, there is always a DE  $\tilde{\Delta}$  which admits  $X$  as a *strong* symmetry and such that  $\Delta$  and  $\tilde{\Delta}$  have the same set of solutions.

**Remark 6.** In a nineteenth-century language, the advantage of knowing symmetries of a differential equation is that the analysis of this, and the search for its solutions, are much easier if one uses the “right” coordinates, i.e. *symmetry-adapted* coordinates – pretty much as analyzing rotationally invariant problems is easier using spherical coordinates.

**2.4. Symmetry of ODEs.** The use of symmetry for ODEs is quite simple; let us focus for simplicity (and for ease of comparison in the case of  $\lambda$ -symmetries to be considered below) on  $\Delta$  a *scalar* ODE of order  $N$ ,

$$F(x, u, \dots, u_{(N)}) = 0. \quad (13)$$

Suppose we were able to determine a Lie-point symmetry  $X = \xi \partial_x + \varphi \partial_u$  for it, and say it is a strong symmetry (if not we can use the CDW theorem and consider the equivalent equation  $\tilde{\Delta}$ , see above); outside singular points of  $X$ , we can pass to coordinates  $(y, v)$  such that  $X = \partial_v$  (flow box theorem [3, 4]); but if in these coordinates  $X$  is written in this way, its prolongation will be  $X^{(N)} = \partial_v$ , as follows immediately from (10).

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<sup>2</sup>More general classes of symmetry can be (and indeed are, in the literature) considered, but we will only consider these.

The equation  $\Delta$  will be written in the new coordinates in some different way, i.e.  $\Delta$  now reads as

$$G(y, v, \dots, v_{(N)}) = 0 ; \quad (14)$$

however the fact that it is invariant under  $X^{(N)}$  and the peculiar form of  $X^{(N)}$  in these coordinates imply that  $G$  does not depend on  $v$ , i.e. we actually have

$$G(y, v_{(1)}, \dots, v_{(N)}) = 0 . \quad (15)$$

It now suffices to make a new change of variables, introducing  $w := v_y$ , to have a differential equation of lower order,

$$H(y, w_{[N-1]}) \equiv G(y, w, \dots, w_{(N-1)}) = 0 . \quad (16)$$

The procedure can be iterated if this has some further symmetry (see also the Remarks below). In this way we obtained a (symmetry) *reduction* of our ODE.

If we are able to solve (16), say to determine a solution  $w = g(y)$ , we can reconstruct a solution  $v = v(y)$  to (15) simply by an integral – in this context one speaks of a *quadrature* – i.e. by

$$v(y) = \int w(y) dy . \quad (17)$$

Inverting the original change of coordinates we obtain a function  $u = u(x)$ , which is a solution to the original equation.

Note that our general notation is redundant for ODEs; in this case all derivatives are w.r.t. a single variable  $x$ , and we can accordingly just write  $u_{(k)}^a = d^k u^a / dx^k$ , and similarly  $\psi_{(k)}^a$  for the coefficient of  $d/du_{(k)}^a$  in  $X^{(m)}$ . The prolongation formula (10) takes then the simpler form

$$\psi_{(k+1)}^a = D_x \psi_{(k)}^a - u_{(k+1)}^a D_x \xi . \quad (18)$$

**Remark 7.** Changing variables from  $(x, u)$  to  $(y, v)$  also means changing the contact forms from  $\omega_j^a = du_j^a - u_{j,i}^a dx^i$  to  $\eta_j^a = dv_j^a - v_{j,i}^a dy^i$ .

**Remark 8.** If the equation has several symmetries, i.e. not only the one we are using for reduction but some other ones as well, it is not guaranteed that these will still be present after the reduction. In general, one can fully use only a (maximal) *solvable* subalgebra of the symmetry algebra of the equation, and this provided the generators are used for reduction in the “right” order; see e.g. [1, 13, 37, 57, 58, 70].

**Remark 9.** On the other hand, the reduced equation could have symmetries which were not present in the original equation. These symmetries appearing upon reduction can be “predicted”, and the features behind their appearance go under the name of “solvable structures”; see e.g. [5, 6, 9, 10, 35, 69] for details.

**Remark 10.** The possibility of effectively operating symmetry reductions as sketchily described above depends on the “invariants by differentiation property”; we refer again to standard texts [1, 13, 37, 57, 58, 70] for details.

**2.5. Symmetry of PDEs.** The use of symmetries in the analysis of PDEs is rather different; actually even the *aim* of using symmetry is different. In fact, for ODEs one can hope of determining the most general solution, or at least (as we have seen above) to determine a reduced equation whose solutions are in correspondence – via a quadrature – to solutions to the original equation.

For nonlinear PDEs looking for the general solution is in general (i.e. except for integrable equations) a hopeless task, and one should instead aim at determining at least some solutions. Again the parallel with the familiar case of rotational symmetries makes things quite clear: one looks first for *symmetric solutions*, and such solutions can be determined by solving a (usually) simpler equation, i.e. one depending on less variables. E.g., rotationally invariant solutions depend just on the radial coordinate  $r$ , and hence are determined by an ODE. In the case of a nonlinear equation this will in general be a nonlinear ODE, and its solution can still be rather hard, but we definitely face a simpler problem than the original one – and correspondingly if we completely solve it, we have only a partial solution to the original one.

Thus, while in the ODE case we were looking for new coordinates in which the symmetry vector field  $X$  was along one of the *dependent* coordinates, in the PDE case we want new coordinates in which the symmetry vector field  $X$  (or vector fields  $X_i$ ) is (are) along one (several) of the *independent* coordinates  $y^i$ . We will then look for solutions  $v = f(y)$  which are invariant under the  $X_i$  ( $i = 1, \dots, r$ ), i.e. which do not depend on the  $(y^1, \dots, y^r)$ ; correspondingly we will have to solve a PDE for  $v$  being a function of the  $(y^{r+1}, \dots, y^n)$  variables, i.e. in less independent variables than the original one.

**Remark 11.** From the geometric point of view, an  $X$ -invariant solution  $u = f(x)$  is a section  $\gamma_f \in \Sigma(M, \pi_0, B)$  such that  $\gamma_f^{(n)} \in S_\Delta$  (which ensures it is a solution) and also such that  $X(\gamma_f) = 0$ , which of course also implies  $X^{(n)}(\gamma_f^{(n)}) = 0$ . If we consider a different vector field  $\tilde{X}$  such that  $\tilde{X} - X = \mu X$  on  $\text{Ker}(X)$  (here  $\mu$  is a smooth function on  $M$ ), such solutions will also be  $\tilde{X}$ -invariant.

**3. Simple twisted symmetries.** In recent years, starting from the seminal work of Muriel and Romero in 2001 [44, 45] (see also [46, 47, 48, 49, 50, 51, 52, 53, 54, 55]), several kinds of *twisted symmetries* have been considered in the literature [21, 22].

The name originated from the fact here one considers a Lie-point vector field  $X$  in  $M$ , but the prolongation operation is deformed in a way which depends on an auxiliary object. In different realizations this can be a scalar function ( $\lambda$ -symmetries [44, 45]), a matrix-valued one form satisfying the horizontal Maurer-Cartan equations – i.e. a set of matrices satisfying a compatibility condition ( $\mu$ -symmetries [16]), or a matrix acting in an auxiliary space ( $\sigma$ -symmetries [17]).<sup>3</sup>

It should also be stressed that twisted symmetries are more easily used for *higher order* differential equations (ordinary or partial), while the case of first order equations is in some sense degenerate from this point of view, and present several additional problems.

**3.1.  $\lambda$ -symmetries.** The first type of twisted symmetries to be introduced was  $\lambda$ -symmetries<sup>4</sup>. This (originally) considered scalar ODEs of any order, and the name refers to the auxiliary  $C^\infty$  function  $\lambda(t, x, \dot{x})$  defining the twisted prolongation,

<sup>3</sup>It should be said that actual “twisting” only occurs in the latter cases, not for  $\lambda$ -symmetries, but I find it convenient to use this collective name [21, 22].

<sup>4</sup>The name  $C^\infty$  symmetries also appears in the literature.

which in this case is called  $\lambda$ -prolongation. In fact, this is recursively defined as

$$\begin{aligned}\psi_{(k+1)}^a &= D_x \psi_{(k)}^a - u_{(k+1)}^a D_x \xi + \lambda \left( \psi_{(k)}^a - u_{(k)}^a \xi \right) \\ &= (D_x + \lambda) \psi_{(k)}^a - u_{(k+1)}^a (D_x + \lambda) \xi.\end{aligned}\quad (19)$$

We will denote the  $\lambda$ -prolongation of order  $k$  of the vector field  $X$  in  $M$  as  $X_\lambda^{(k)}$ .

The vector field  $X$  in  $M$  is said to be a  $\lambda$ -*symmetry* of the equation  $\Delta$  (of order  $k$ ) if

$$X_\lambda^{(k)} : S_\Delta \rightarrow \mathbf{T} S_\Delta . \quad (20)$$

Note that in general a vector field is a  $\lambda$ -symmetry of a given equation *only* for a specific choice of the function  $\lambda$ .

**Lemma 4.** In general, the commutator of the  $\lambda$ -prolongations of two vector fields  $X, Y$  in  $M$  is *not* the  $\lambda$ -prolongation of their commutator, i.e. if  $Z = [X, Y]$  then (in general, for  $\lambda \neq 0$ )

$$\left[ X_\lambda^{(n)}, Y_\lambda^{(n)} \right] \neq Z_\lambda^{(n)} . \quad (21)$$

**Proof.** Consider e.g.  $X = x\partial_u$ ,  $Y = u\partial_x$ ; in this case  $Z = [X, Y] = x\partial_x$ , and  $\delta := [X_\lambda^{(1)}, Y_\lambda^{(1)}] - Z_\lambda^{(1)} = x\lambda + (u - xu_x)\lambda_{u_x}$ .

**Lemma 5.** *The IBDP holds for  $\lambda$ -prolonged vector fields.*

**Proof.** See e.g. [44, 45], or [21].

**Remark 12.** Lemma 5 makes  $\lambda$ -symmetries “as useful as standardly prolonged ones”, as we will see in a moment.

**Remark 13.** It was pointed out by Pucci and Saccomandi [63] that  $\lambda$ -prolonged vector fields can be characterized as the *only* vector fields in  $J^k M$  with the property that their integral lines are the same as the integral lines of some vector field which is the standard prolongation of some vector field in  $M$ . This remark was fully understood only some time after their paper, and was the basis for much of the following developments, discussed below.

**3.2.  $\mu$ -symmetries.** The  $\lambda$ -prolongation is specifically designed to deal with ODEs (or systems thereof); a generalization of it aiming at tackling PDEs (or systems thereof) is the  $\mu$ -prolongation. This can of course also be applied to ODEs and Dynamical Systems.

**3.2.1. PDEs.** Now the relevant object is not a single matrix, but an array of matrices  $\Lambda_i$ , one for each independent variable. These are better encoded as a ( $GL(n, \mathbf{R})$ -valued) *horizontal one-form*

$$\mu = \Lambda_i(x, u, u_x) dx^i . \quad (22)$$

The matrices  $\Lambda_i$  should satisfy a compatibility condition, i.e.

$$D_i \Lambda_j - D_j \Lambda_i + [\Lambda_i, \Lambda_j] = 0 ; \quad (23)$$

this is immediately recognized as the *horizontal Maurer-Cartan equation*, or equivalently as a *zero-curvature condition* for the connection on  $TU$  identified by

$$\nabla_i = D_i + \Lambda_i . \quad (24)$$

If  $\mu$  satisfies (23), we can define  $\mu$ -prolongations in terms of a modified prolongation formula, called of course  *$\mu$ -prolongation formula* (and which represents now an actual twisting of the familiar prolongation operation):

$$\begin{aligned}\psi_{J,i}^a &= D_i \psi_J^a - u_{J,k}^a D_i \xi^k + (\Lambda_i)_b^a (\psi_J^b - u_{J,k}^b \xi^k) \\ &= (D_i I + \Lambda_i)_b^a \psi_J^b - u_{J,k}^b (D_i I + \Lambda_i)_b^a \xi^k .\end{aligned}\quad (25)$$

We will denote the  $\mu$  prolongation (of order  $k$ ) of the vector field  $X$  in  $M$  as  $X_\mu^{(k)}$ . The vector field  $X$  in  $M$  is said to be a  $\mu$ -*symmetry* of the equation  $\Delta$  (of order  $k$ ) if

$$X_\mu^{(k)} : S_\Delta \rightarrow \mathbb{T} S_\Delta . \quad (26)$$

Note that in general a vector field is a  $\mu$ -symmetry of a given equation *only* for a specific choice of the one-form  $\mu$ .

**Remark 14.** In  $\lambda$ -prolongations the prolongation operation is modified, but it acts separately on the different vectorial components in  $TU$  (and in  $TU_J$ ). In  $\mu$ -prolongations, instead, the different vector components of  $TU$  (and of  $TU_J$ ) are instead “mixed” by the prolongation operation.

**Remark 15.** It is known that  $\mu$ -symmetries (and hence  $\lambda$ -symmetries, which are a special case of the latter) are related to *nonlocal* symmetries; we will not discuss this relation here [8, 49, 54].

**3.2.2. ODEs.** In the case of ODEs one just replaces the scalar function  $\lambda : J^1 M \rightarrow \mathbf{R}$  with a *matrix* function  $\Lambda : J^1 M \rightarrow \mathbf{Mat}(n)$  (more generally,  $\Lambda : J^1 M \rightarrow TU$ ) and define a “ $\Lambda$ -prolongation” (which is just a special case of  $\mu$ -prolongation, for  $\mu = \Lambda dx$ )

$$\begin{aligned}\psi_{(k+1)}^a &= D_x \psi_{(k)}^a - u_{(k+1)}^a D_x \xi + \Lambda_b^a (\psi_{(k)}^b - u_{(k)}^b \xi) \\ &= (D_x I + \Lambda)_b^a \psi_{(k)}^b - u_{(k+1)}^b (D_x I + \Lambda)_b^a \xi .\end{aligned}\quad (27)$$

In this ODE case we just have  $\mu = \Lambda dx$  (only one component), and (23) is identically satisfied.

**Remark 16.** The IBDP property is in general not holding for  $\mu$ -prolonged vector fields, not even in the ODEs framework; the exception is the case where the  $\Lambda_i$  are diagonal matrices.

**3.2.3. Recursion formula.** The  $\mu$ -prolongation  $X_\mu^{(k)}$ , which we will now write in components as  $X_\mu^{(k)} = \xi^i \partial_i + (\psi_J^a)_{(\mu)} \partial_a^J$ , of a vector field  $X$  in  $M$  is defined through (25); however in some cases and applications it is relevant to characterize these in terms of the difference

$$F_J^a := (\psi_J^a)_\mu - (\psi_J^a)_0 . \quad (28)$$

It can be shown [16, 30] that the  $F_J^a$  satisfy the recursion relation

$$F_{J,i}^a = \delta_b^a \left[ D_i (\Gamma^J)_c^b \right] (D_i Q^c) + (\Lambda_i)_b^a \left[ (\Gamma^J)_c^b (D_J Q^c) + D_J Q^b \right] , \quad (29)$$

where we have written

$$Q^a = \varphi^a - u_i^a \xi^i , \quad (30)$$

and the  $\Gamma^J$  are certain matrices whose detailed expression can be computed [16, 30] but is not essential.

**Remark 17.** With the notation (30), the set  $I_X$  of  $X$ -invariant functions is characterized by  $Q^a|_{I_X} = 0$ . It follows then from (29) that  $X_\mu^{(k)}$  coincides with  $X_0^{(k)}$  on  $I_X$ .

**4. Collective twisted symmetries:  $\sigma$ -symmetries.** Let us consider the vector structure in  $TU$  and more generally in  $T^kU$ . We have seen that in  $\lambda$ -prolongations different components (in terms of this structure) of a vector field “do not mix” under the prolongation operation, while in  $\mu$ -prolongations they do indeed “mix”.

As mentioned above, Pucci and Saccomandi [63] observed that (in the scalar case)  $\lambda$ -prolongations are the *only* vector fields in  $J^kM$  which have the same characteristics as *some* standardly prolonged vector field.

One can extend this approach to *distributions* generated by sets – in involution *à la Frobenius* – of standardly prolonged vector fields, and wonder of there is some deformation of the prolongation operations such that a set of vector fields obtained by this generate the same distribution in  $J^kM$  as some other set of vector fields prolonged in the standard way.

This problem was tackled by Cicogna *et al.* in a series of papers [17, 18, 19, 20] and the answer is that the most general class of systems with this property is provided by so called  $\sigma$ -prolonged sets of vector fields. Note that here the deformation of the prolongation operation involves *sets* of vector fields, and not a single one. We also stress that we are working in the frame of ODEs, hence with only one independent variable  $x$ .<sup>5</sup>

Given vector fields  $X_\alpha$  ( $\alpha = \dots, r$ ) in  $M$ , written in local coordinates as

$$X_\alpha = \xi_\alpha \partial_x + \varphi_\alpha^a \partial_a, \quad (31)$$

and satisfying the Frobenius involution relations

$$[X_\alpha, X_\beta] = f_{\alpha\beta}^\gamma X_\gamma \quad (32)$$

with  $f_{\alpha\beta}^\gamma : J^1M \rightarrow \mathbf{R}$  smooth functions on  $J^1M$ ,

The  $\sigma$ -prolonged vector fields  $Y_\alpha$  on  $J^kM$  are then written as

$$Y_\alpha = \xi_\alpha \partial_x + (\psi_k^a)_\alpha \partial_a^k \quad (33)$$

where  $(\psi_0^a)_\alpha = \varphi_\alpha^a$  and

$$(\psi_{k+1}^a)_\alpha = (D_x(\psi_k^a)_\alpha - u_{k+1}^a D_x \xi_\alpha) + \sigma_\alpha^\beta ((\psi_k^a)_\beta - u_{k+1}^a \xi_\beta). \quad (34)$$

**Lemma 6.** Let  $X_\alpha$  satisfy (32), and *assume* their  $\sigma$ -prolongations  $Y_\alpha$  are in involution. Then the set  $\{Y_\alpha\}$  has the IBDP property<sup>6</sup>.

**Remark 18.** For fields  $X_\alpha$  satisfying (32) and  $Y_\alpha$  their  $\sigma$ -prolongations, in general,

$$[Y_\alpha, Y_\beta] \neq f_{\alpha\beta}^\gamma Y_\gamma. \quad (35)$$

However, the  $Y_\alpha$  can happen to be still in involution, or to be embedded in set of vector fields in involution of non-maximal rank. This is why in Lemma 6 the involution of the  $Y_\alpha$  has to be assumed.

<sup>5</sup>Actually  $\sigma$ -prolongations and symmetries can also be defined in the framework of PDEs (they go then under the name of  $\chi$ -symmetries), but here we are mainly concerned with Dynamical Systems.

<sup>6</sup>We specify that in this case the IBDP property should be meant as follows: if  $\eta$  and  $\zeta_{(k)}$  are independent common differential invariants for all of the  $Y_\alpha$ , then so are the  $\zeta_{(k+1)} := (D_x \zeta_{(k)}) / (D_x \eta)$ .

**Remark 19.** While the  $\mu$ -prolongations mix different vector component of the same vector field, here corresponding components of different vector fields are mixed. For  $r = 1$  we are back to the case of  $\lambda$ -prolongations.

**Remark 20.** If  $\sigma = \text{diag}(\lambda_1, \dots, \lambda_n)$  is a diagonal matrix (but not a multiple of the identity) we have different vector fields undergoing  $\lambda$ -prolongations with different functions  $\lambda_i$ . In the case of  $\sigma = \lambda I$  we are back to the case of  $\lambda$ -prolongations (in general, applied to a set of vector fields in multidimensional space).

**5. The use of twisted symmetries.** We have so far discussed the definition of different types of twisted prolongations and hence of twisted symmetries. We would now like to discuss how these are applied in the study of differential equations. In doing this one should distinguish between ODEs and PDEs, recalling that – as also stressed above – the very aim of symmetry theory is different in these two contexts.

We will always assume that the vector field  $X$  is a twisted symmetry (of different types) of the equations under study.

**5.1. The use of  $\lambda$ -symmetries.** If  $X$  is a symmetry for an equation  $\Delta$  of order  $n$ , this means that  $\Delta$  can be written in terms of the differential invariants for  $X_\mu^{(n)}$ . On the other hand, as we have seen above (Lemma 5),  $\lambda$ -prolonged vector fields enjoy the IBDP. This implies that passing to  $\lambda$ -symmetry-adapted coordinates, one can indeed rewrite the equation in terms of differential invariants of order zero and one and their total derivatives, implementing the reduction procedure sketched in Section 2.4.

In other words, the usual symmetry reduction algorithm can be applied also in the case of  $\lambda$ -symmetries, which are as useful as standard ones in the study of ODEs.<sup>7</sup>

**5.2. The use of  $\mu$ -symmetries.** As mentioned above,  $\mu$ -symmetries were thought for being applied in the study of PDEs. Here the key fact is (29) (see also Remark 17); in fact, in studying PDEs by the symmetry approach one is aiming at determining invariant solutions, and (29) shows that when restricting to  $X$ -invariant solutions it makes no difference to consider standard prolongations or  $\mu$ -prolongations. This also entails that we can use the same methods and techniques familiar from the case  $X$  is a standard symmetry also in the case  $X$  is a  $\mu$ -symmetry (and in general not a proper symmetry).

**Remark 21.** This also shows that  $\mu$ -symmetries of a given equation are strong candidates for being also (standard) conditional symmetries [40, 62], or partial symmetries [14], for the same equation.

**Remark 22.** The situation is different in the case of ODEs (this case was studied by Cicogna<sup>8</sup> [11, 12]). In this case one can proceed pretty much as in the standard reduction procedure up to a (relevant) feature: that is, the reconstruction equation, which in the standard case amounts to a quadrature, is now a proper differential equation, and its solution may very well be very hard, or turn out to be impossible. See [11, 12] for details.

<sup>7</sup>We mention in passing that  $\lambda$ -symmetries have also been used for the reduction of *discrete* equations [38, 39]; this lies outside our scope here.

<sup>8</sup>He speaks in this case of  $\rho$ -symmetries, the  $\rho$  standing for “reducing”, see below.

**5.3. The use of  $\sigma$ -symmetries.** It follows immediately by Lemma 6 that  $\sigma$ -symmetries can also be used to reduce (systems of) ODEs in the same way and with the same procedure as in the standard case. Once again, the key fact is that this standard reduction procedure [1, 13, 37, 57, 58, 70] is actually based on the IBDP.

It should be stressed, however, that in this case there is a further condition to be satisfied, i.e. that the  $Y_\alpha$  are (or can be completed in a nontrivial way to a system) in involution.

**Remark 23.** In this context, it should also be mentioned that determining  $\sigma$ -symmetries is in general a nontrivial task (recall that the determination of standard symmetries is often computationally demanding, but always algorithmic); but when one considers as differential equation a perturbation of a system for which symmetries are known,  $\sigma$ -symmetries can be sought for as deformation of the symmetries for the unperturbed system; see [19] (and Section 9 below) for details.

**6. Twisted symmetries and gauge transformations.** It appears that twisted symmetries are related to gauge transformations, and indeed the operators  $\nabla_i = D_i + \Lambda_i$  appearing in  $\mu$ -prolongations look very much like a covariant derivative. We will now make this relation more precise. For this, it is convenient to consider just vertical vector fields, including evolutionary representatives of general vector fields in  $M$ .

**Lemma 7.** Let  $X = Q^a \partial_a$  and  $\tilde{X} = \tilde{Q}^a \partial_a$  be vertical vector fields on  $(M, \pi_0, B)$ ,  $A : M \rightarrow \mathbf{Mat}(\mathbf{R}, q)$  (with  $q = \dim(U)$ ) a nowhere zero smooth matrix function, and  $\tilde{Q}^a = A^a_b Q^b$ . Then

$$A \left( X_\mu^{(n)} \right) = \tilde{X}_0^{(n)} ; \quad \mu = (DA) A^{-1} . \quad (36)$$

**Remark 24.** The relation between  $X_\mu^{(n)}$  and  $\tilde{X}_0^{(n)}$  in (36) should be meant as follows: if  $X_\mu^{(n)} = \psi_J^a \partial_a^J$ , with  $\psi_0^a = Q^a$  and the  $\psi_J^a$  for  $1 \leq |J| \leq n$  obtained by the  $\mu$ -prolongation formula, and  $\tilde{X}^{(n)} = \tilde{\psi}_J^a \partial_a^J$  with  $\tilde{\psi}_0^a = \tilde{Q}^a$  and the  $\tilde{\psi}_J^a$  for  $1 \leq |J| \leq n$  obtained by the standard prolongation formula, then the relation

$$\tilde{\psi}_J^a = A^a_b \psi_J^b \quad (37)$$

holds for any  $a$  and  $J$ ,  $0 \leq |J| \leq n$ .

**Remark 25.** The relations stated by Lemma 7 can be encoded in a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{A} & \tilde{X} \\ \downarrow \mu\text{-prol} & & \downarrow \text{prol} \\ X_\mu^{(n)} & \xrightarrow{A} & \tilde{X}_0^{(n)} \end{array} \quad (38)$$

where  $A$  and  $\mu$  are related by  $\mu = (DA)A^{-1}$ .

**Remark 26.** The Lemma 7 is *not* stating that any  $\mu$ -prolonged vector field is obtained as the gauge transformed of a standardly prolonged one; this relation only holds for vertical vector field. If  $X = Q^a \partial_a$  is the evolutionary representative of a generic vector field  $X_g = \xi^i \partial_i + \varphi^a \partial_a$ , hence  $Q^a = \varphi^a - u_i^a \xi^i$ , its components  $Q^a$  satisfy the relations  $(\partial Q^a / \partial u_i^b) = -\delta_b^a \xi^i$ . These are obviously not satisfied in general by the components  $\tilde{Q}^a = A^a_b Q^b$  of  $\tilde{X}$  (now  $(\partial \tilde{Q}^a / \partial u_i^b) = -A_b^a \xi^i$ ), hence we cannot

interpret the gauge-transformed vector field as the evolutionary representative of a vector field in  $M$  [22].

**Remark 27.** The previous Remark also means that the connection between  $\mu$ -prolongations and gauge transformations is only transparent when we consider the action of vector fields on  $\Sigma(M)$ , the set of sections on  $M$ . It also explains why we can have twisted symmetries in cases (that is, for equations) where there are no standard symmetries.

**Remark 28.** More details on the interrelations between (different kinds of) twisted symmetries and gauge transformations are provided in a number of papers [23, 24, 25].

**7. Twisted symmetries and Frobenius theory.** The existence of a relation between twisted symmetries and gauge transformations was more and less evident since the introduction of  $\lambda$ -symmetries by Muriel and Romero, and so the generalization of  $\lambda$ -symmetries to  $\mu$ -symmetries was, in this sense, not surprising.

It was much less obvious that symmetries and twisted symmetries could be generalized in a different direction, focusing on *sets* of vector fields rather than on single ones<sup>9</sup>. As already mentioned, the key step in this direction was provided by Pucci and Saccomandi [63], who stressed the symmetry relation involves the integral lines of (prolongations of) symmetry vector fields, not the way in which the flow defined by the vector field travels on these.

The geometrical idea behind  $\sigma$ -symmetries is that (standard) symmetry vector fields for the equation  $\Delta$  of order  $n$  are the vector fields on  $M$  whose (standard) prolongation to  $J^n M$  belongs to the *distribution* tangent to  $S_\Delta$  in  $TJ^n M$ . Focusing on the distribution – rather on the single vector fields, i.e. the “usual” generators of the distribution – has an obvious consequence: we can change the generators of the distribution.

In particular, if we are dealing with ODEs, we would like to be able to change the generators of the distribution (to have more freedom), but at the same time be sure that the key tool for ODE reduction, i.e. the IBDP, is still at work.

The idea of  $\sigma$  symmetries is exactly this: the  $\sigma$ -prolongation is the more general way of twisting prolongation by mixing *different* vector fields in such a way that the IBDP still holds, and hence so that twisted symmetries can be of use for the reduction of ODEs.

**Lemma 8.** Let  $\mathcal{X} = \{X_1, \dots, X_r\}$  be a set of vector fields on  $M$ ; and let the vector fields  $\mathcal{Y} = \{Y_1, \dots, Y_r\}$  on  $J^n M$  be their  $\sigma$ -prolongation. Consider also  $A : M \rightarrow \text{Mat}(\mathbf{R}, q)$  (where  $q = \dim(U)$ ) a nowhere singular matrix function on  $M$ , such that  $\sigma = A^{-1}(D_x A)$ ; and the set  $\mathcal{W} = \{W_1, \dots, W_r\}$  of vector fields on  $M$  given by  $W_\alpha = A_\alpha^\beta X_\beta$ , with  $\mathcal{Z} = \{Z_1, \dots, Z_r\}$  on  $J^n M$  their standard prolongation. Then,  $Z_\alpha = A_\alpha^\beta Y_\beta$ .

**Remark 29.** The relation between  $Z_\alpha$  and  $W_\beta$  in Lemma 8 should be meant as the one in Lemma 7.

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<sup>9</sup>One speaks therefore of “collective” twisted symmetries. Actually here we will only deal with the case of ODEs and Dynamical Systems ( $\sigma$ -symmetries), rather than general PDEs ( $\chi$ -symmetries). For the latter, the interested reader is referred to [19, 22].

**Remark 30.** The relations stated by Lemma 8 can be encoded in a commutative diagram:

$$\begin{array}{ccc} \{X_i\} & \xrightarrow{A} & \{W_i\} \\ \downarrow \sigma\text{-}prol & & \downarrow prol \\ \{Y_i\} & \xrightarrow{A} & \{Z_i\} \end{array} \quad (39)$$

where  $A$  and  $\sigma$  are related by  $\sigma = A^{-1}(DA)$ .

**Remark 31.** Traditionally, in the symmetry analysis of differential equations one focuses on the Lie algebra structure of symmetry vector fields. Passing to consider Frobenius reduction means one is instead focusing on the *Lie module* structure.<sup>10</sup>

**Remark 32.** The determination of standard symmetries is algorithmic for equations of order  $n \geq 2$ , but is specially difficult for first order equations – even more so for first order ODEs, i.e. Dynamical Systems – albeit in general we always have infinitely many symmetries in this case. In the case of Dynamical Systems, an interesting possibility was noted by Cicogna [19]: if we consider the perturbations of a symmetric Dynamical Systems,  $\sigma$ -symmetries can be looked for by building  $\sigma$  as a perturbation to the identity matrix. See Section 9 below.

**Remark 33** The possibility of using sequentially different symmetries for Frobenius reduction rests – like in the case of standard reduction – on a suitable involution structure; that is, we should have a solvable Lie-module.

**Remark 34.** More details on  $\sigma$ -prolongations and symmetries, including their geometrical meaning, is provided in the series of paper [17, 18, 19, 20]; see also [22].

**8. Twisted symmetries and variational problems.** The theory of twisted symmetries developed mainly referring to ODEs and evolution PDEs. But we know that in Physics a special role is played by problems admitting a *variational formulation*. In this case, the relation between symmetries and conservation laws is embodied by the classical Noether theorem [2, 36, 57]. It is thus natural to wonder if there is a version of Noether theorem applying to *twisted* symmetries.

Only partial results exist in this direction; these are concerned with  $\lambda$ -symmetries [56, 65] and with  $\mu$ -symmetries [15], while it seems no result is available dealing with  $\sigma$ -symmetries.

**8.1. Variational problems and  $\lambda$ -symmetries.** The relation between  $\lambda$ -symmetries and Euler-Lagrange equations has been considered in a by now classical work of Muriel, Romero and Olver [56]; lately new results in this direction have been announced by Ruiz, Muriel and Olver [65].

**8.1.1. Single  $\lambda$ -symmetry of a variational problem.** We consider variational problems defined by a Lagrangian density  $L$ , hence by

$$S[u] = \int L(x, u_{[n]}) dx ; \quad (40)$$

here  $x \in \mathbf{R}$ ,  $u \in \mathbf{R}$ . To this problem are associated the Euler-Lagrange equations

$$E[L; u] = \sum_{k=0}^n (-D_x)^k \left( \frac{\partial L}{\partial u_k} \right) = 0 . \quad (41)$$

<sup>10</sup>Note that while some equations (in particular all equations which are linear or can be linearized by a change of variables) have an infinite dimensional Lie algebra of symmetries, their set of symmetries is always finite dimensional as a Lie module.

A vector field  $X = \xi(x, u)\partial_x + \varphi(x, u)\partial_u$  is a standard variational symmetry [57] if there is a function  $F : J^n M \rightarrow \mathbf{R}$  such that

$$X^{(n)}(L) + L(D_x \xi) = D_x F. \quad (42)$$

This definition is generalized by saying that  $X$  is a *variational  $\lambda$ -symmetry* if there is a function  $F : J^n M \rightarrow \mathbf{R}$  such that

$$X_{(\lambda)}^{(n)}(L) + L[(D_x + \lambda)\xi] = (D_x + \lambda)F. \quad (43)$$

If  $X$  is a variational  $\lambda$  symmetry for  $L$ , it is such also for  $\tilde{L} = L + (D_x f)$ , for any  $f : J^n M \rightarrow \mathbf{R}$ , i.e. for any equivalent Lagrangian [56].

Variational  $\lambda$ -symmetries lead to reduction of order for the variational problem in the same way as standard variational symmetries. More precisely, Muriel, Romero and Olver prove the following result (Theorem 1 in [56]).

**Lemma 8.** Let  $S[u]$  as in (40) be an  $n^{\text{th}}$ -order variational problem with Euler-Lagrange equation  $E[L; u] = 0$  of order  $2n$ . Let  $X$  be a variational  $\lambda$ -symmetry, where  $\lambda : J^1 M \rightarrow \mathbf{R}$  is smooth. Then there exists a variational problem

$$\widehat{S}[w] = \widehat{L}(\tilde{x}, w_{[n-1]})d\tilde{x} \quad (44)$$

of order  $n - 1$ , with Euler-Lagrange equation  $E[\widehat{L}; w] = 0$  of order  $2n - 2$ , such that a  $(2n - 1)$ -parameter family of solutions of  $E[L; u] = 0$  can be found by solving a first-order equation from the solutions of the Euler-Lagrange reduced equation  $E[\widehat{L}; w] = 0$ .

As for the Noether theorem, this essentially follows (for standard symmetries) from

$$X^{(n)}(L) = Q E[L] + D_x F \quad (45)$$

where  $F$  is some function  $F : J^n M \rightarrow \mathbf{R}$ , and  $Q = \varphi - u_x \xi$  is the characteristic of the (evolutionary representative of)  $X$ .

In the case of  $\lambda$ -prolongations, one can prove [56] that there is some  $F$  such that

$$X_{\lambda}^{(n)}(L) = Q E[L] + (D_x + \lambda)F. \quad (46)$$

Then the following result (which is Theorem 2 in [56]) follows

**Lemma 10.** Let  $X$  be a variational  $\lambda$ -symmetry of the variational problem (40), and  $Q$  the characteristic of  $X$ . Then there exists  $P[u] : J^n M \rightarrow \mathbf{R}$  such that

$$Q E[L] = (D_x + \lambda)P. \quad (47)$$

**Remark 35.** While standard variational symmetries of a variational problem are symmetries of the corresponding Euler-Lagrange equations,  $\lambda$ -symmetries of the variational problem are in general *conditional* symmetries of the Euler-Lagrange equations [56] (see also Remark 17 in this respect).

**Remark 36.** If  $X$  is a variational  $\lambda$ -symmetry of (40), and  $P[u]$  is the functional given in Lemma 10, then  $X$  is a  $\lambda$ -symmetry of the equation  $P[u] = 0$ . The reduction of this equation through  $X$  is (up to multipliers) the reduced equation of the EulerLagrange equation corresponding to  $X$ , according to Lemma 9 [56].

8.1.2. *Multiple  $\lambda$ -symmetry of a variational problem.* In more recent work, Ruiz, Muriel and Olver [65] studied variational problems systems which admit several  $\lambda$ -symmetries  $X_i$ , where for each of them a *different* function  $\lambda_i$  defines  $\lambda$ -prolongation.

They considered in particular the case of two such  $\lambda$ -symmetries  $\{X, Y\}$ , subject to the “solvability condition”

$$\left[ X_{\lambda_1}^{(n)}, Y_{\lambda_2}^{(n)} \right] = h X_{\lambda_1}^{(n)}. \quad (48)$$

Then the  $\lambda$ -symmetries can be used (in the proper order!) to perform *two* symmetry reductions of the variational problem.<sup>11</sup> In particular, one can prove [65] that:

**Lemma 11.** Let (40) be an  $n$ -th order variational problem with Euler-Lagrange equation  $E[L; u] = 0$  of order  $2n$ . Let  $(X_1, \lambda_1), (X_2, \lambda_2)$  be variational  $\lambda$ -symmetries that form a solvable pair, as in (48). Then there exists a variational problem  $\widehat{S} = \int \widehat{L}[x, z_{n-2}] dx$  of order  $n-2$  such that a  $(2n-2)$ -parameter family of solutions of  $E[L; u] = 0$  can be reconstructed from the solutions of the associated  $(2n-4)$ -th order Euler-Lagrange equation  $E[\widehat{L}; z] = 0$  by solving two successive first order ordinary differential equations.

**Remark 37.** Albeit  $\lambda$ -prolongations with different functions  $\lambda_i$  for different vector fields fit within  $\sigma$ -prolongations – see in particular Remark 20 – it should be stressed that variational problems have never been studied in that frame. Thus Lemma 11 calls for a full study of Frobenius reduction in the variational context.

8.2. **Variational problems and  $\mu$ -symmetries.** A different approach to twisted symmetries (in particular,  $\mu$ -symmetries) in variational problems was considered by Cicogna *et al.* [15]. In this work they show that  $\mu$ -symmetries are associated to so called  $\mu$ -conservation laws<sup>12</sup> and in the end, for variational problems with a single independent variable (dynamical variational problems) and  $\Lambda = \lambda I$ , to conditionally conserved quantities [41, 61, 64, 66, 67].

These are quantities such that only *some* of their level set correspond to invariant manifolds – while for proper conserved quantities *all* the level set are dynamically invariant; note that the results is strictly related to the one mentioned in Remark 35.

There is a different way of looking at variational problems, in particular dynamical variational problems, with  $\mu$ -symmetries; this descends from the relation between  $\mu$ -prolongations and gauge transformations.

If we think of such a problem as arising – through a gauge transformation – from one in which the  $\mu$ -prolonged vector field was prolonged in the standard way, i.e. perform the needed inverse gauge transformation (see section 6), we should think that the original variational problem was a different one, and Euler-Lagrange equations were also different. In particular, the role of  $D_i$  in the Euler-Lagrange operator would be taken by  $\nabla_i = D_i + \Lambda_i$ . Thus, e.g., in the case of Mechanics the  $\mu$ -Euler-Lagrange equations read

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = (\Lambda^T)_i^j \frac{\partial L}{\partial \dot{q}^j}. \quad (49)$$

<sup>11</sup>It appears that the result is extended to  $k$   $\lambda_i$ -symmetries with a suitable solvability condition, i.e. generating a solvable Lie module.

<sup>12</sup>A conservation law is a relation of the type  $D_i \cdot \mathbf{P}^i = 0$  for some vector  $\mathbf{P}$ ; a  $\mu$ -conservation law reads  $\text{Tr}(\nabla_i \cdot \mathbf{P}^i) = 0$ , with  $\nabla_i = D_i + \Lambda_i$ .

One can then show [15] that<sup>13</sup>:

**Lemma 12.** If  $L$  is a first-order Lagrangian admitting the vector field  $X = \varphi^a \partial_a$  as a  $\mu$ -symmetry, then  $\mathbf{P}$  of components  $P^i = \varphi^a \pi_a^i$  (where  $\pi_a^i = \partial L / \partial u_a^i$ ) defines a *standard* conservation law,  $D_i P^i = 0$ , for the flow of the associated  $\mu$ -Euler-Lagrange equations.

**9. Twisted symmetries and perturbations of Dynamical Systems.** As mentioned above (see Remarks 23 and 32),  $\sigma$ -symmetries turn out to be specially suited for the investigation of perturbations of symmetric Dynamical Systems. It should be stressed that in this case also the determination of  $\sigma$ -symmetries (which is, as for all types of twisted symmetries, a non-algorithmic task) turns out to be facilitated.

The type of result one can obtain in this direction is illustrated by the following result (Theorem 4 in [19]):

**Lemma 13.** Let the dynamical system

$$\frac{dx^i}{dt} = f^i(x) \quad (50)$$

admit the vector fields  $X_\alpha = \varphi_\alpha^i \partial_i$  as standard symmetries, and let these span a Lie algebra,

$$[X_\alpha, X_\beta] = c_{\alpha\beta}^\gamma X_\gamma. \quad (51)$$

Consider moreover the vector fields  $Y_\alpha$  in  $JM$  obtained as  $\sigma$ -prolongations of the  $X_\alpha$  with

$$\sigma_\alpha^\beta = c_{\alpha\gamma}^\beta F^\gamma + X_\alpha(F^\beta). \quad (52)$$

Then: (i) The  $Y_\alpha$  are in involution and satisfy the same commutations relations as the  $X_\alpha$ ; (ii) any dynamical system of the form

$$\frac{dx^i}{dt} = f^i(x) + \sum_{\alpha=1}^r F^\alpha(x) \varphi_\alpha^i(x) \quad (53)$$

admits the set of  $X_\alpha$  as  $\sigma$ -symmetries – with  $\sigma$  given by (52) – and hence can be reduced via these.

**Remark 38.** The form (53) of systems which can be dealt with in this way may seem too specific, but it includes at least one relevant class, i.e. that of systems in *normal form* [4, 13]. In fact, let  $f(x) = Ax$  with  $A$  a non-degenerate semi-simple matrix; then we consider  $\varphi_\alpha(x) = B_\alpha x$  with matrices such that  $[A, B_\alpha] = 0$  (these obviously form a Lie algebra  $\mathcal{G}$ , and  $B_0 = A$  is always in the set). Then the polynomial vector fields which admit  $X_0 = (Ax)^i \partial_i$  as symmetry are just those written in the form (53), with  $F^\alpha(x)$  generators for the ring of  $X_0$ -invariant functions. See [19] for details and examples, as well as [29, 31, 32] for related matters.

**Remark 39.** Orbital reduction of dynamical systems [34, 72, 73] can be dealt with in a similar manner; we will not discuss this here.

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<sup>13</sup>For further details, proofs and examples, see [15].

**10. Conclusions.** The theory of twisted symmetries of differential equations has been created in 2001; it passed from a smart observation by his creators [44, 45] to a coherent set of results, and from an analytic formulation to a geometrical one. In particular, in the course of this travel several relations with *gauge transformations* and with the Frobenius theory of vector fields have been uncovered, and it has been realized how twisted symmetries become rather natural if one looks not at the standard theory of (Lie) reduction, but at *Lie-Frobenius reduction* for differential equations.

It has also been realized that twisted symmetries – in the form of “perturbed prolongations” – can be used to study perturbations of symmetric equations and in particular symmetric Dynamical Systems; this part of the theory definitely awaits further developments.

Similarly, the study of twisted symmetries (and their use) for variational problems is in its first steps, and is worth receiving further attention.

Albeit we have not touched this topic at all, I would like to mention also that in the recent wave of interest for symmetries of *stochastic* differential equations (see [27] and references therein) there is not yet any work studying the role (if any) of twisted symmetries in that context.

I hope these pages can help attracting mathematicians to this nice and promising field; it would be even nicer if our young friend Juergen Scheurle himself could contribute to the topic.

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