Symmetry and integrability for stochastic differential equations

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We discuss the interrelations between symmetry of an Ito stochastic differential equations (or systems thereof) and its integrability, extending in party results by R. Kozlov [*J. Phys. A* **43** (2010) & **44** (2011)]. Together with integrability, we also consider the relations between symmetries and reducibility of a system of SDEs to a lower dimensional one. We consider both "deterministic" symmetries and "random" ones, in the sense introduced recently by Gaeta and Spadaro [*J.Math. Phys.* **58** (2017)].

I. INTRODUCTION

It is well known that symmetry methods are among the most powerful tools for the study of nonlinear deterministic differential equations [3, 7, 25, 26, 29]. It is thus natural to think they could be useful also in the study of *stochastic* differential equations (SDEs).

This observation is of course not new, and indeed there is by now a substantial literature devoted to symmetries of SDEs (see [12] for an extended reference list). In a first phase the efforts focused on determining the proper – that is, useful – definition of symmetry for a SDE and hence the *determining equations*; there is now a consensus on what this suitable definition is (see below). The second and crucial phase is of course to understand how these can be *used* in the study of SDEs.

The main idea – paralleling the approach for deterministic equations – is to use symmetry-adapted coordinates to implement a *reduction* of the SDE. It should be stressed that here one should think of the decomposition of a given system into a smaller (lower dimensional) one plus one or more *reconstruction equations*, as in the case of deterministic Dynamical Systems; we are here going to study systems of Ito equations, which are the stochastic counterpart of first order ODEs, i.e. indeed Dynamical Systems.

A key result in this direction was obtained by Kozlov [18] for scalar SDEs, showing that if such an SDE possesses a symmetry of a certain type ("simple" symmetries, to be defined below), then it can be integrated; the theorem is actually constructive, i.e. the symmetry determines a change of variables which allows for explicit integration of the SDE (see below for details).

In the case of systems of SDE, Kozlov's approach [19, 20] shows that a symmetry of the appropriate type implies that the system can be "partially integrated" (in a sense to be made precise below).

Kozlov approach is based – like analogous results for deterministic equations – on changes of variables to use favorable properties of symmetry-adapted coordinates; it is essential that symmetries are preserved under changes of coordinates. In a companion paper [13] we have argued that while this fact is entirely obvious for deterministic equations, preservation of symmetries for Ito stochastic equations cannot be given for granted. The reason is that Ito equations are not geometrical objects, and indeed do not transform in the "usual" way (i.e. under the familiar chain rule) under changes of coordinates: in fact, they transform according to the Ito rule. However it turns out that there is a rather ample class of symmetries which *are* preserved under changes of coordinates, and these include the kind of symmetries relevant for Kozlov's theory; the reader is referred to our work [13] for a discussion, while the results relevant to our present subject are recalled – after discussing in Section II some general features of symmetries of SDEs – in Section III.

Equipped with these preliminary results, we pass to study the relation between symmetry and integration – or at least reduction – for (Ito) SDEs. For what concerns a single symmetry, the canonical result in the literature, already recalled above, is due to Kozlov [18] and says that the existence of a symmetry of a given type is a *sufficient* condition for the integrability of an Ito scalar SDE. We will show that the condition is not only sufficient but also *necessary*, see Section IV.

We will then pass to consider the case of multi-dimensional systems of SDEs with a single symmetry and discuss the reduction which is possible in this case (see Section V); this will also give an idea of the kind of results which can be obtained in the general case.

Finally (in Section VI) we will consider the case of a n-dimensional system of Ito SDEs with a q-dimensional solvable symmetry algebra (the restriction to considering *solvable* symmetry algebras descends from well known results in the study of deterministic equations and their symmetry reduction [3, 7, 25, 26, 29]), and discuss how the symmetry allows for a multiple-stages reduction of the system; in general this is actually setting the system in a "partially triangular"

form, with an m-dimensional (with $m < n$) system and a number ($q = n - m$) of "reconstruction equations". Here again we will compare our results with those obtained by Kozlov [19, 20].

So far we have considered maps acting by transformation of the dependent variables, possibly depending on time; we will also show (in Section VII, with examples in Section VIII) that the results discussed so far can be extended to the case of *random maps* [4, 5, 15], i.e. transformations depending on the Wiener processes entering in the equations under study. In this case the transformation is subject to certain restrictions to guarantee we remain within the category of Ito equations, and a number of other details should be controlled. For this setting, we will consider only *scalar* equations, postponing the study of systems to a future contribution. It will turn out that it is possible to obtain a nearly full extension – at least for scalar equations – of Kozlov theory to the setting of random maps and random symmetries; the only difference being that now an additional compatibility condition (68) involving the coefficients of the Ito equation and of the symmetry vector field should be satisfied.

We also provide two Appendices; the first shows that albeit one could apparently extend Kozlov's theory to other types of integrable SDEs (separable ones), this framework is already contained in the "standard" case; in the second we discuss when one has infinitely many (simple) random symmetries for a given scalar Ito equation.

Finally, it should be mentioned that the literature also considers (systems of) higher order SDEs and their symmetries [12, 28]; we will not consider these, neither other possible generalizations of the setting (i.e., that of systems of Ito equations) we have chosen to study. Similarly, when dealing with deterministic equations one also considers more general classes of symmetries than Lie-point ones [3, 7, 25, 26, 29]; but we will not consider these here.

II. SYMMETRY OF STOCHASTIC DIFFERENTIAL EQUATIONS

In this note, by "differential equation" (DEs) we will always mean possibly a vector one, i.e. a system of differential equations, unless the contrary is explicitly stated (in this case we speak of a scalar equation).

Given a deterministic DE Δ of order n in the basis manifold M (this include dependent and independent variables), it is well known that this is identified with its *solution manifold* S_Δ in the Jet bundle $J^n M$. A Lie-point symmetry (generator) is then a vector field X on M which, once *prolonged* to the vector field $X^{(n)}$ on $J^n M$, is tangent to $S_\Delta \subset J^n M$ [3, 7, 25, 26, 29].

Both submanifolds and vector fields are geometrical object; they are defined independently of any coordinate system, and it is thus entirely obvious that tangency relations – hence in particular X being a symmetry for Δ – hold independently of coordinates. This also means that symmetries are preserved under changes of coordinates.

Let us consider an (Ito) stochastic differential equation (SDE) [4, 9, 10, 16, 17, 24]

$$dx^i = f^i(x, t)\, dt + \sigma^i{}_j(x, t)\, dw^j \; ; \tag{1}$$

in (1) and below, $t \in R$, $x \in R^n$ (the x^i can be thought as local coordinates on a smooth manifold, as we take care of distinguishing covariant and contravariant indices albeit never introducing explicitly a metric), f and σ are smooth vector and matrix functions of their arguments, and w^j ($j = 1, ..., m$) are independent standard Wiener processes.

For Ito equations, a *geometrical* interpretation of the equation is missing, and one is forced to resort to a purely *algebraic* notion of symmetry. Actually, not only a geometrical interpretation is missing, but this is not possible: in fact Ito equations and more generally Ito differentials do not transform in the "usual" way, i.e. according to the familiar chain rule, under changes of coordinates; they transform instead according to the Ito rule, and this is inherent to their very nature [4, 9, 10, 16, 17, 24, 30] (see [27], chapter 5, for a Physics point of view).

Remark 1. The situation is different for Stratonovich SDEs; actually the main motivation for their introduction was precisely to have SDEs transforming "nicely" under changes of coordinates. For these a geometrical description is indeed possible, and hence symmetries are trivially preserved under any change of coordinates (see also the detailed discussion in [13]). On the other hand, Stratonovich equations present several problems from the probabilistic point of view; see e.g. [4, 9, 10, 16, 17, 24, 30]. It should be stressed that albeit there is a correspondence between Ito and Stratonovich equations (which is not entirely trivial, see e.g. the discussion in [30]), there is not an identity between general symmetries – even of deterministic type [32] – of an Ito equation and those of the corresponding Stratonovich one. It is instead known that such an identity holds for (both deterministic [32] and random [13]) *simple* symmetries (see again [13] for discussion and examples). Note that previous work by one of the present authors [12, 15] contains wrong statements in this respect (originating in a trivial mistake in the definition of the Ito Laplacian, as discussed in [13]). ⊙

Remark 2. It should be stressed that, in view of the non-covariance of Ito equations, and of the purely algebraic definition of symmetries for them (see e.g. the review [12] for details) it is not at all guaranteed *a priori* that symmetries will be preserved under changes of coordinates. An analysis of this problem was provided recently [13],

and it turned out that the symmetries which are relevant for integrability and/or reduction are preserved. Such results will be recalled below, and are based on the relation with symmetries of the corresponding Stratonovich equation, see Remark 1. ⊙

One could, in principles, consider general mappings and vector fields in the (x, t, w) space – albeit with some restrictions if these have to make physical sense, see e.g. [12, 15]. That is, we could generally speaking consider maps

$$(x, t; w) \ \to \ \big(\widetilde{x}(x, t, w), \widetilde{t}(x, t, w); \widetilde{w}(x, t, w) \big) \ .$$

Within these, we will denote by **simple** the maps (and vector fields) which act only on the spatial variables, leaving the time t (and the Wiener processes) unchanged. We will also denote as **deterministic** the maps (and vector fields) which only depend on x (and possibly t), as opposed to the **random** ones (to be considered in subsection II B below), in which dependence on the Wiener processes w is also present.

Within our present context, we are interested only in *simple* maps. The reason for this is that only these are relevant in Kozlov theory, as discussed in his works [18–20] and as will be recalled below. This simplifies considerably our task.

A. Simple deterministic maps & symmetries

Let us first consider a (simple) smooth[**?**] map

$$(x, t) \ \to (\widetilde{x}, t) \ , \quad \widetilde{x} = \widetilde{x}(x, t) \ ; \tag{2}$$

this induces a map $dx \to d\widetilde{x}$ and hence a map on SDEs (1); in particular if (2) is (locally) inverted to give

$$x \ = \ \Phi(\widetilde{x}, t) \ , \tag{3}$$

then by Ito formula

$$dx^i \ = \ \left(\frac{\partial \Phi^i}{\partial \widetilde{x}^j} \right) d\widetilde{x}^j \ + \ \left(\frac{\partial \Phi}{\partial t} \right) dt \ + \ \frac{1}{2} \left(\frac{\partial^2 \phi^i}{\partial \widetilde{x}^j \partial \widetilde{x}^k} \right) \widetilde{\sigma}^j_{\ \ell} \widetilde{\sigma}^k_{\ \ell} \, dt \ ; \tag{4}$$

similarly the functions $f^i(x, t)$ and $\sigma^i_{\ j}(x, t)$ are mapped into functions $\widetilde{f}^i(\widetilde{x}, t)$ and $\widetilde{\sigma}^i_{\ j}(\widetilde{x}, t)$. In this way, (1) is mapped into a new Ito equation

$$d\widetilde{x}^i \ = \ \widehat{f}^i(\widetilde{x}, t) \, dt \ + \ \widehat{\sigma}^i_{\ j}(\widetilde{x}, t) \, dw^j \ ; \tag{5}$$

note that here the $\widehat{f}$ and $\widehat{\sigma}$ take into account not only the change of their variables, but also the contribution arising from dx expressed in the new variables via the Ito formula.

We say that (2) is a **symmetry** for (1) if (5) is identical to (1), i.e. if the $(n + m \cdot n)$ conditions

$$\widehat{f}^i(x, t) \ = \ f^i(x, t) \ , \quad \widehat{\sigma}^i_{\ j}(x, t) \ = \ \sigma^i_{\ j}(x, t) \tag{6}$$

are satisfied identically in (x, t).

We are specially interested in the case where the map (2) is a near-identity one, $\widetilde{x}^i = x^i + \varepsilon(\delta x)^i$ and can hence be seen as the infinitesimal action of a vector field X,

$$X \ = \ \varphi^i(x, t) \frac{\partial}{\partial x^i} \ \equiv \ \varphi^i(x, t) \, \partial_i \ . \tag{7}$$

(Note that X has no component along the t variable; this corresponds to the restriction made above, see eq.(2), for simple symmetries.) In this case we speak of a *Lie-point* (simple) symmetry.

We obtain then easily (the reader is referred to [14] for details and explicit computations) the *determining equations for (simple, deterministic) Lie-point symmetries of SDEs*:

$$\varphi^i_t \ + \ f^j \, (\partial_j \varphi^i) \ - \ \varphi^j \, (\partial_j f^i) \ = \ -\frac{1}{2} \, (\Delta \varphi^i) \ , \tag{8}$$

$$\sigma^j_{\ k} \, (\partial_j \varphi^i) \ - \ \varphi^j \, (\partial_j \sigma^i_{\ k}) \ = \ 0 \ . \tag{9}$$

Here and below Δ denotes the *Ito Laplacian*, which in general (for functions possibly depending also on the w^k, see next subsection) is defined as

$$\Delta f \ = \ \delta_{ik} \, \left[\left(\frac{\partial^2 f}{\partial w^i \partial w^k} \right) \ + \ \sigma^j_{\ i} \sigma^m_{\ k} \left(\frac{\partial^2 f}{\partial x^j \partial x^m} \right) \ + \ 2 \, \sigma^j_{\ i} \left(\frac{\partial^2 f}{\partial x^j \partial w^k} \right) \right] \ . \tag{10}$$

B. Simple random symmetries

We can consider more general transformations; in particular one can consider maps and vector fields which depend on the Wiener processes realizations w^k. This approach (actually in a more general setting) was first considered by Arnold and Imkeller [4, 5] in the context of *normal forms* for SDEs; random symmetries of SDEs are considered e.g. in [15] (see also [12]), to which we refer for explicit computations and details.

For what concerns our discussion here, we consider *simple random maps*,

$$(x, t; w) \rightarrow (\widetilde{x}, t; w) , \quad \widetilde{x} = \widetilde{x}(x, t; w) ; \tag{11}$$

and the corresponding symmetries, i.e. *simple random symmetries*. These are the vector fields

$$X = \varphi^i(x, t; w) \, \partial_i \tag{12}$$

leaving the equation (1) invariant

It is shown in [15] that, using the notation, to be used routinely in the following,

$$\widehat{\partial}_k := \frac{\partial}{\partial w^k} , \tag{13}$$

the *determining equations for simple random Lie-point symmetries of a SDE* (1) read

$$(\partial_t \varphi^i) + f^j (\partial_j \varphi^i) - \varphi^j (\partial_j f^i) = -\frac{1}{2} (\Delta \varphi^i) , \tag{14}$$

$$(\widehat{\partial}_k \varphi^i) + \sigma^j{}_k (\partial_j \varphi^i) - \varphi^j (\partial_j \sigma^i{}_k) = 0 . \tag{15}$$

III. PRESERVATION OF SIMPLE SYMMETRIES OF ITO EQUATIONS

Ito equations are not geometrical objects. In fact, under changes of coordinates they do not transform in the "usual" way, i.e. under the chain rule, but in their own way, i.e. under the Ito rule.

This means that symmetries are defined in an *algebraic* rather than *geometrical* way; and, at difference with deterministic differential equations, it is not obvious that if we consider an Ito equation E, a vector field X which is a symmetry for E, and a change of coordinates (mapping the equation E into an equation $\widetilde{E}$), the vector field X will also be a symmetry for the transformed equation $\widetilde{E}$. This point was raised and discussed in a recent paper of ours [13].

The situation is however quite different – and similar to the familiar one for deterministic equations – if one considers, as in Kozlov theory, only *simple symmetries*.

Lemma 1. *Simple (random or deterministic) symmetries of an Ito equation are preserved under changes of coordinates.*

Proof. See [13]. $\triangle$

IV. DETERMINISTIC SYMMETRY AND INTEGRABILITY FOR A SCALAR ITO SDE

We will consider the following result, which in its essential part is due to Kozlov:

Theorem 1. *The SDE*

$$dy = \widetilde{f}(y, t) \, dt + \widetilde{\sigma}(y, t) \, dw \tag{16}$$

can be transformed by a deterministic map into

$$dx = f(t) \, dt + \sigma(t) \, dw , \tag{17}$$

and hence explicitly integrated, if and only if *it admits a simple deterministic symmetry.*

If the generator of the latter is

$$X = \varphi(y, t) \, \partial_y , \tag{18}$$

then the change of variables $y = F(x, t)$ transforming (16) into (17) is the inverse to the map $x = \Phi(y, t)$ identified by

$$\Phi(y, t) \;=\; \int \frac{1}{\varphi(y, t)} \, dy \; . \tag{19}$$

Proof. Consider a (scalar) SDE of the form (17). This is obviously and elementarily integrable; moreover it admits the Lie symmetry generator

$$X \;=\; \partial_x \; . \tag{20}$$

If we operate a change of variable of the form

$$x \;=\; \Phi(y, t) \tag{21}$$

(note we are leaving t and $w = w(t)$ untouched), the Ito formula gives

$$dx \;=\; (\partial_t \Phi) \, dt \;+\; (\partial_y \Phi) \, dy \;+\; \frac{1}{2} \, (\partial_y \Phi)^2 \, \sigma^2(t) \, dt \; . \tag{22}$$

Putting together this and (17) we get

$$\Phi_y \, dy \;=\; \left[f(t) \;-\; \Phi_t \;-\; \frac{1}{2} \, \Phi_{yy} \, \widetilde{\sigma}^2(t) \right] dt \;+\; \sigma(t) \, dw \; . \tag{23}$$

Now we just note that $\Phi_y \neq 0$, or (21) would not be a proper change of variable, so that we can divide by Φ_y. Thus (17) reads, in terms of the new variable, precisely as (16), where we have written

$$\widetilde{f}(y, t) \;:=\; \frac{1}{\Phi_y} \left[f \;-\; \Phi_t \;-\; \frac{1}{2} \, \Phi_y^2 \, \sigma^2 \right] \; ; \tag{24}$$

$$\widetilde{\sigma} \;:=\; \frac{\sigma}{\Phi_y} \; . \tag{25}$$

On the other hand, in the new coordinates the vector field X reads

$$X \;=\; \frac{\partial y}{\partial x} \, \partial_y \;=\; \frac{\partial F(x, t)}{\partial x} \, \frac{\partial}{\partial y} \;=\; \frac{1}{\Phi_y} \, \partial_y \; . \tag{26}$$

Following the discussion in Sect.III, we are guaranteed by Lemma 1 that (26) is a symmetry for (16).

In view of the general nature of (21), we have thus shown that if the equation (16) can be mapped via a simple change of variables of the form

$$y \;=\; F(x, t) \; , \tag{27}$$

to the equation (17), then necessarily it enjoys the symmetry (26). The required change of variables (27) is just the inverse to (21), i.e. these satisfy

$$\Phi[F(x, t), t] = x \; ; \quad F[\Phi(y, t), t] = y \; . \tag{28}$$

It just remains to identify F in terms of the coefficient $\varphi(y, t)$ in (18). Comparing the latter and (26) we immediately have

$$\varphi(y, t) \;=\; \frac{1}{\Phi_y(y, t)} \; , \tag{29}$$

which shows that (19) holds and identifies $\Phi(y, t)$ in terms of $\varphi(y, t)$. As we need the change of variables leading from y to x, we just need the change of variable $y = F(x, t)$ inverse to Φ, as stated in (28). This completes the proof in one direction, i.e. shows that indeed symmetry is a necessary condition for the equation (16) to be integrable.

In order to show that the condition is also sufficient for the integrability of (16), it suffices to perform the change of variables, thus obtaining (17) which is obviously integrable. $\triangle$

Remark 3. The Theorem is due to Kozlov [18], who stated it in one direction only – i.e. identifying the symmetry condition as a *sufficient* one to guarantee integrability – and gave it in constructive form, i.e. identifying the change of variables which makes the equation explicitly integrable. Our contribution here is to show that this condition is not only sufficient but also *necessary*. ⊙

Remark 4. One could think of extending the Kozlov theorem to more general situations: in fact, stochastic equations of the form (17) are not the only ones which can be integrated, and one could e.g. consider also those of the form $dx = \beta(x)\,[f(t)\,dt \,+\, \sigma(t)\,dw]$. This can be done, but actually gives no new result. The discussion of this fact is given in A. ⊙

Remark 5. The most relevant feature of Kozlov theorem is of course that it is *constructive*: once we have determined the simple symmetry (if any) of the equation, it suffices to invert (29), which of course yields (19), to explicitly get the required change of variables and explicitly integrate the equation. ⊙

Remark 6. The function $\Phi(y,t)$ is determined by (19) up to an "integration constant" which is actually an arbitrary function of t; it is clear from (24), (25) that the only effect of this would be to add a function of t alone to $F(y,t)$ – or, seen from the other end of the procedure, to $f(t)$ – while leaving $S(y,t)$ and $\sigma(t)$ unchanged; thus with no loss of generality for what concerns the integration of the SDE under study. (Similar considerations would also apply for the results in the next section V, and will not be repeated there.) ⊙

Example 1. The Ito equation[**?**]

$$dy \;=\; \left[e^{-y} \;-\; (1/2)\,e^{-2y}\right]\,dt \;+\; e^{-y}\,dw \tag{30}$$

admits the vector field $X = e^{-y}\partial_y$ as a (Lie-point) symmetry generator. By the change of variables $x = \exp[y]$ the vector field reads $X = \partial_x$, and the initial equation (30) reads

$$dx \;=\; dt \;+\; dw\;. \tag{⊙}$$

Example 2. The (quite involved) equation

$$dy \;=\; \frac{e^{-t}\,(1+y^2)^2}{8y^3}\,\left(-4y^2 \,+\, e^t(3y^4 + 2y^2 - 1)\right)\,dt \;-\; \frac{(1+y^2)^2}{2y}\,dw \tag{31}$$

admits the vector field $X = -[(1+y^2)^2/(2y)]\partial_y$ as a symmetry. Correspondingly, passing to the variable $x = 1/(1+y^2)$, we get $X = \partial_x$ and the equation (31) just reads

$$dx \;=\; e^{-t}\,dt \;+\; dw\;. \tag{⊙}$$

V. DETERMINISTIC SYMMETRY AND REDUCTION FOR A SYSTEM OF ITO SDES

In the previous section we have considered reduction of a scalar Ito equation under a simple Lie-point symmetry. In view of the discussion given there it is not surprising that the same result – with some obvious changes – also holds when we consider a system of Ito SDEs with a simple symmetry. In this section we will discuss in detail such an extension. We will again show that Kozlov's sufficient conditions are also necessary.

We thus consider a system of n SDEs for $x^1(t),...,x^n(t)$, which enjoys a symmetry of an appropriate type, and want to reduce this to a system of $n-1$ equations for $y^1(t),...,y^{n-1}(t)$; the processes $y^i(t)$ will be defined in terms of the $x^i(t)$. We have the following result.

Theorem 2. *Consider the system (1), with $i = 1,...,n$. The* necessary and sufficient *condition for the existence of a simple deterministic non-degenerate change of variables (2) which brings the system into the form*

$$dy^i \;=\; g^i(y^1,...,y^{n-1},t)\,dt \;+\; \rho^i{}_k(y^1,...,y^{n-1},t)\,dw^k(t) \quad (i=1,...,n) \tag{32}$$

is that (1) admits a simple deterministic Lie-point symmetry (7).

Proof. We will proceed substantially in the same way as in the proof for Theorem 1. We start from the "reduced" system (32), which is clearly invariant under the action of the vector field $X = (\partial/\partial y^n)$; we consider a general

(obviously, invertible) change of variables, i.e. we bring it to its general form in arbitrary coordinates. So we consider the change of coordinates $(y^1, ..., y^n) \to (x^1, ..., x^n)$ defined by

$$y^i = F^i(x, t) , \tag{33}$$

with inverse

$$x^i = \Phi^i(y, t) . \tag{34}$$

Under this change of variables the system (32) becomes (using Ito rule)

$$dy^i = dF^i(x, t) = \frac{\partial F^i}{\partial x^k} dx^k + \frac{\partial F^i}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F^i}{\partial x^k \partial x^\ell} dx^k dx^\ell . \tag{35}$$

Let us focus on the last term, and more specifically on the factor $dx^k dx^\ell$; using (34) and again Ito rule,

$$\begin{aligned} dx^k dx^\ell &= \frac{\partial \Phi^k}{\partial y^m} \frac{\partial \Phi^\ell}{\partial y^s} dy^m dy^s + o(dt) \\ &= \frac{\partial \Phi^k}{\partial y^m} \frac{\partial \Phi^\ell}{\partial y^s} \rho^m_p \rho^s_q dw^p dw^q + o(dt) \\ &= \frac{\partial \Phi^k}{\partial y^m} \frac{\partial \Phi^\ell}{\partial y^s} \rho^m_q \rho^s_q dt + o(dt) . \end{aligned}$$

Inserting this into (35), the latter reads

$$dy^i = \frac{\partial F^i}{\partial x^k} dx^k + \frac{\partial F^i}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F^i}{\partial x^k \partial x^\ell} \left(\frac{\partial \Phi^k}{\partial y^m} \frac{\partial \Phi^\ell}{\partial y^s} \rho^m_q \rho^s_q \right) dt . \tag{36}$$

These allow to deduce the SDEs satisfied by the process in terms of the new (i.e. the x) variables; in fact, using also (32), the above relation (36) is rewritten as

$$\frac{\partial F^i}{\partial x^k} dx^k = \left[g^i - \frac{\partial F^i}{\partial t} - \frac{1}{2} \frac{\partial^2 F^i}{\partial x^k \partial x^l} \left(\frac{\partial \Phi^k}{\partial y^m} \frac{\partial \Phi^\ell}{\partial y^s} \rho^m_q \rho^s_q \right) \right] dt + \rho^i_k dw^k(t) . \tag{37}$$

In order to obtain a system of the form (1), we need to invert the matrix $(\partial F^i / \partial x^k)$, that is the Jacobian matrix of F. This inverse is of course the Jacobian for the inverse change of coordinates, i.e. the Jacobian for Φ; in fact

$$\delta^i_k = \frac{\partial y^i}{\partial y^k} = \frac{\partial F^i[\Phi(y, t), t]}{\partial x^k} = \frac{\partial F^i}{\partial x^l} \frac{\partial \Phi^l}{\partial y^k} . \tag{38}$$

By acting on (37) from the left with the matrix $(\partial F / \partial y) = (\partial F / \partial x)^{-1}$, we get

$$\begin{aligned} dx^k &= \frac{\partial \Phi^k}{\partial y^i} \left[g^i - \frac{\partial F^i}{\partial t} - \frac{1}{2} \frac{\partial^2 F^i}{\partial x^m \partial x^l} \left(\frac{\partial \Phi^k}{\partial y^m} \frac{\partial \Phi^\ell}{\partial y^s} \rho^m_q \rho^s_q \right) \right] dt + \frac{\partial \Phi^k}{\partial y^i} \rho^i_\ell dw^\ell \\ &:= f^k(x, t) dt + \sigma^k_\ell(x, t) dw^\ell . \end{aligned} \tag{39}$$

In the last step we have simply defined

$$f^i(x, t) := g^i - \frac{\partial F^i}{\partial t} - \frac{1}{2} \frac{\partial^2 F^i}{\partial x^m \partial x^l} \left(\frac{\partial \Phi^k}{\partial y^m} \frac{\partial \Phi^\ell}{\partial y^s} \rho^m_q \rho^s_q \right) , \tag{40}$$

$$\sigma^i_k(x, t) := \frac{\partial \Phi^i}{\partial y^j} \rho^j_k ; \tag{41}$$

here all the quantities which are function of y should be thought as function of x through (33).

Let us now turn our attention to symmetries. The original system (32) is by construction invariant under $X = (\partial / \partial y^n)$. In the x variables this vector field is described by

$$X = \left(\frac{\partial x^i}{\partial y^n} \right) \frac{\partial}{\partial x^i} = \frac{\partial \Phi^i}{\partial y^n} \frac{\partial}{\partial x^i} := \varphi^i(x, t) \partial_i . \tag{42}$$

It should be stressed that when considering (possibly time-dependent) vector fields acting in $\mathbf{R}^n$, we are dealing with the x^i, y^i as coordinates in $\mathbf{R}^n$, not with the stochastic processes $x^i(t)$, $y^i(t)$ defined by SDEs (32), (1); thus we do *not* have to use the Ito formula.

As discussed in Section III, it is not guaranteed *apriori* that symmetries present in one system of coordinates will still be present in another one; however as we deal with simple symmetries we are guaranteed by Lemma 1 we still have X – now given as (42) – as a symmetry.

We have thus determined the most general SDE (1) which can be obtained from (32) by a change of variables; these are identified by (40), (41). Reversing our point of view, we have shown that all the SDEs (1) which can be reduced to the form (32) by a change of variables are characterized by f, σ as given in (40), (41). Moreover these admit by construction the Lie-point symmetry (42). This shows that indeed the presence of such a symmetry is a necessary condition for the equation (1) to be reducible to the form (32).

Moreover, our discussion shows that there is a simple relation between the symmetry (42) and the change of variables needed to take (1) into the reduced form (32); this is given by

$$\varphi^i = \frac{\partial \Phi^i}{\partial y^n} . \tag{43}$$

In order to show that the presence of such a symmetry (42) for the equation (1) characterized by (40), (41) is also a sufficient condition for its (reducibility to the form (32) and hence) integrability, it suffices to note (as in [18]) that the change of variables (33) with Φ given by (43) produces exactly the required reduction. $\triangle$

Remark 7. If $(y^1(t), ..., y^{n-1}(t))$ are a solution to the autonomous subsystem consisting of the first $n-1$ equations in (32), then $y^n(t)$ is given by

$$y^n(t) = y^n(t_0) + \int_{t_0}^t g^n[y^1(t), ..., y^{n-1}(t); t] \, dt + \int_{t_0}^t \rho^n{}_k[y^1(t), ..., y^{n-1}(t); t] \, dw^k(t) , \tag{44}$$

and is therefore known. In the language used in the symmetry analysis of deterministic equations, this can be seen as a "reconstruction equation", and amounts to a (stochastic) quadrature. $\odot$

Example 3. Let us consider the two-dimensional system (we write all indices as subscripts to avoid confusion)

$$dy_1 = \left(e^{y_1} - \frac{1}{2} e^{-2y_1} \right) dt + e^{-y_1} \, dw_1$$

$$dy_2 = \frac{1}{2} e^{y_2} \left(2e^{y_1} + e^{y_2} + e^{2y_1+y_2} \right) dt - e^{y_1+y_2} \, dw_1 - e^{y_2} \, dw_2 . \tag{45}$$

This admits as symmetry the vector field $X = -e^{y_2} (\partial/\partial y_2)$. Actually, passing to the variables $x_1 = e^{y_1}$, $x_2 = e^{-y_2}$, the initial system (45) is rewritten as

$$dx_1 = x_1^2 \, dt + dw_1$$
$$dx_2 = -x_1 \, dt + x_1 \, dw_1 + dw_2 . \tag{46}$$

In these coordinates, $X = (\partial/\partial x_2)$. $\odot$

VI. MULTIPLE SYMMETRIES AND REDUCTIONS FOR SYSTEMS OF SDES

It is natural to ask now what happens if the system of SDEs (1) admits more than one symmetry, in particular – in view of well known results holding for deterministic equations [3, 7, 25, 26, 29] – what happens if it admits a r-parameter solvable group of symmetry. This was stated in [19, 20] but without a formal proof. We will now give a formal statement and a detailed proof.

Theorem 3. *Suppose the system (1) admits an r-parameter solvable algebra $\mathfrak{g}$ of simple deterministic symmetries, with generators*

$$\mathbf{X}_k = \sum_{i=1}^n \varphi_k^i(x, t) \frac{\partial}{\partial x_i} \qquad k = 1, ..., r, \tag{47}$$

acting regularly with r-dimensional orbits.

Then it can be reduced to a system of $m = (n - r)$ equations,

$$dy^i = g^i(y^1, ..., y^m; t) \, dt + \sigma^i_{\ k}(y^1, ..., y^m; t) \, dw^k \quad (i, k = 1, ..., m) \tag{48}$$

and r "reconstruction equations", the solutions of which can be obtained by quadratures from the solution of the reduced $(n - r)$-order system.

In particular, if $r = n$, the general solution of the system can be found by quadratures.

Proof. We want to proceed by induction on the dimension r of the algebra (for $r = 1$ this theorem obviously reduces to the previous one).

First of all we recall that the hypothesis on the algebraic structure of $\mathfrak{g}$ implies that there exists a chain of algebras $\mathfrak{g}^{(k)}$ ($k = 1, ..., r$) of dimension k such that $\mathfrak{g}^{(0)} \subset \mathfrak{g}^{(1)} \subset \cdots \subset \mathfrak{g}^{(r+1)} = \mathfrak{g}$, and each algebra $\mathfrak{g}^{(k)}$ is a normal subalgebra of $\mathfrak{g}^{(k+1)}$, i.e.

$$[\mathfrak{g}^{(k)}, \mathfrak{g}^{(k+1)}] \subset \mathfrak{g}^{(k)} \ .$$

Suppose now we have an $r = s + 1$-parameter solvable group of symmetries G with Lie algebra $\mathfrak{g}$. Let $\mathbf{X}_{s+1}$ be a generator of $\mathfrak{g}^{(s+1)}$ that does not lie in $\mathfrak{g}^{(s)}$, and consider the one-parameter group G_{s+1} generated by it. Thanks to the solvability condition, the system is invariant under the action of G_{s+1}, since its action coincides to that of the quotient $\mathfrak{g}^{(s+1)}/\mathfrak{g}^{(s)}$. Thus we can invoke Theorem 2 and reduce the system via a change of variable $x \to \widetilde{x}$, getting a "reduced" system of $n - 1$ equations for $\widetilde{x}^1(t), ..., \widetilde{x}^{n-1}(t)$; and a last "reconstruction" equation which amounts to

$$\widetilde{x}^n = \widetilde{x}^n(t_0) + \int_{t_0}^t \overline{f}^n(\widetilde{x}^1, ..., \widetilde{x}^{n-1}, t) \, dt + \int_{t_0}^t \overline{\sigma}^n_k(\widetilde{x}^1, ..., \widetilde{x}^{n-1}, t) \, dw^k(t) \ . \tag{49}$$

The key observation now is that it has been proven (Lemma 1) that the symmetries are maintained under changes of variables; thus we know for sure that the remaining $s = r - 1$ symmetries of the original system are still symmetries of the reduced one (once expressed in the new variables).

This allows to iterate the previous step and carry on the procedure described above, using the remaining s vector fields one by one. After performing all the allowed r steps we will have a reduced system of $n - r$ equations for $(\widehat{x}^1(t), ..., \widehat{x}^{n-r}(t))$, and r "reconstruction" equations generalizing (49). Note that if we are able to obtain a solution to the reduced system, solutions to the full system require to solve the reconstruction equation in the "proper" order, i.e. the one dictated by the solvable structure of the symmetry algebra (to obtain solutions to the original equations we will of course also have to invert all the change of coordinates performed at each step in the reduction procedure).

Note also that if $r = n$, at the last step (after using $r - 1$ symmetries) the reduced system will amount to a single equation, with a symmetry of the form considered in Theorem 1; using this we will get, as in Theorem 1, a single SDE with coefficients depending only on the variable t, hence an integrable SDE. Solutions to the full systems are then obtained by solving the reconstructions equations. $\triangle$

Remark 8. We stress that the solvability hypothesis was essential to guarantee that the actions of the quotients $G^{(k+1)}/G^{(k)}$ (where $G^{(k)}$ is the Lie group generated by $\mathfrak{g}^{(k)}$, and so on) were symmetries of the system, according to the fact that if a normal subgroup H of the group G is a symmetry of a system, then G itself is also a symmetry if and only if the system reduced under H admits the quotient G/H as a symmetry group. $\odot$

Example 4. We will just refer the reader to Example 4.2 in Kozlov paper [18]. It is shown there that any equation of the form

$$\begin{aligned} dx_1 &= (a_1 + b_{11}x_1 + b_{12}x_2) \, dt + s_{11} \, dw_1 + s_{12} \, dw_2 \\ dx_2 &= (a_2 + b_{21}x_1 + b_{22}x_2) \, dt + s_{21} \, dw_1 + s_{22} \, dw_2 \end{aligned} \tag{50}$$

with a_i, b_{ij} and s_{ij} real constant and satisfying the "full rank condition", which in this case just reads

$$\texttt{Det} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \neq 0 \ ,$$

admits symmetries

$$X = \varphi_1(t) \, \frac{\partial}{\partial x_1} + \varphi_2(t) \, \frac{\partial}{\partial x_2} \ ,$$

with φ_i arbitrary smooth functions. Looking at two vector fields of the form

$$X_i \;=\; A_i(t)\,\frac{\partial}{\partial x_1} \;+\; B_i(t)\,\frac{\partial}{\partial x_2}\;,$$

with $\Delta := A_1 B_2 - A_2 B_1 \neq 0$ to ensure independence, the corresponding change of variables is

$$y_1 \;=\; \frac{B_2(t)\,x_1 \;-\; A_2(t)\,x_2}{A_1(t)\,B_2(t) \;-\; A_2(t)\,B_1(t)}\;;\;\; y_2 \;=\; \frac{-B_1(t)\,x_1 \;+\; A_1(t)\,x_2}{A_1(t)\,B_2(t) \;-\; A_2(t)\,B_1(t)}\;.$$

With this, the system (50) is mapped into

$$dy_1 \;=\; \left(\frac{a_1 B_2 - a_2 A_2}{A_1 B_2 - A_2 B_1}\right)\,dt \;+\; \left(\frac{B_2 s_{11} - A_2 s_{21}}{A_1 B_2 - A_2 B_1}\right)\,dw_1 \;+\; \left(\frac{B_2 s_{12} - A_2 s_{22}}{A_1 B_2 - A_2 B_1}\right)\,dw_2$$

$$dy_2 \;=\; \left(\frac{a_2 A_1 - a_1 B_1}{A_1 B_2 - A_2 B_1}\right)\,dt \;+\; \left(\frac{A_1 s_{21} - B_1 s_{11}}{A_1 B_2 - A_2 B_1}\right)\,dw_1 \;+\; \left(\frac{A_1 s_{22} - B_1 s_{12}}{A_1 B_2 - A_2 B_1}\right)\,dw_2\;,$$

which is indeed of the form

$$dy^i \;=\; F^i(t)\,dt \;+\; S^i{}_k(t)\,dw^k$$

and hence integrable. The reader is referred to [18] for full detail. $\quad\odot$

VII. RANDOM MAPS AND KOZLOV THEOREM

We have so far considered changes of variables of the form (2); we will now consider a more general class of transformations, i.e. *random maps* [5].

Lemma 1 does still guarantee that symmetries are preserved under (simple) random maps, but in this case the discussion becomes more involved, and we will limit to consider simple random maps, see (11) and (12), and the framework of *scalar* Ito equations, i.e. the analogue of the situation discussed in Section IV (systems will be discussed in a forthcoming contribution).

The difficulty here lies in that we are not guaranteed a simple random map (11) will take an Ito equation into an equation of Ito type. In fact, let us consider a scalar Ito equation (16) and assume the existence of a symmetry in the form $X = \varphi(y, t, w)\partial_y$. If we change coordinates passing to

$$x \;=\; \Phi(y, t, w) \;=\; \int \frac{1}{\varphi(y, t, w)}\, dy \tag{51}$$

(this integral is defined up to an additive function $\beta(t, w)$, which is a "constant of integration" in this framework), the vector field X is mapped again into $X = \partial_x$, and Lemma 1 guarantees this is a symmetry of the transformed equation, which we write as $dx = f dt + \sigma dw$. Knowing that $X = \partial_x$ is a symmetry for this equation guarantees – in view of the determining equations (14), (15) – that $(\partial f/\partial x) = 0$, $(\partial \sigma/\partial x) = 0$; but now the coefficients f and σ could depend on w as well, in which case the transformed equation is not even an Ito one.

Remark 9. On the other hand, *if* the equation (16) can be obtained from an integrable equation, i.e. one with $\widetilde{f} = \widetilde{f}(t)$, $\widetilde{\sigma} = \widetilde{\sigma}(t)$, via a simple change of coordinates $y = \Theta(x, t, w)$, then necessarily the map identified by Φ given above – or more precisely by *some* Φ as above (the one which defines the change of variable inverse to the one defined by Θ), i.e. for some choice of the function $\beta(t, w)$ playing the role of integration constant – takes (16) back to the original one, with no dependence of the coefficients on w. $\quad\odot$

In this context, we will find useful to consider how a simple random change of variables

$$x \;=\; \Phi(y, t; w) \tag{52}$$

acts on the the general (formal and) not necessarily Ito equation

$$dx \;=\; f(t, w)\,dt \;+\; \sigma(t, w)\,dw\;. \tag{53}$$

Lemma 2. *Under the simple random change of variables (52), the general formal equation (53) is mapped into a (formal) equation of the type*

$$dy \; = \; \widehat{F}(y,t,w) \, dt \; + \; \widehat{S}(y,t,w) \, dw \tag{54}$$

with coefficients given by

$$\widehat{F}(y,t,w) \; = \; \frac{f - \Phi_t - (1/2)\Delta\Phi}{\Phi_y} \; , \quad \widehat{S}(y,t,w) \; = \; \frac{\sigma - \Phi_w}{\Phi_y} \; . \tag{55}$$

Proof. This follows by a standard computation. We start from (53), and operate with the general change of variables (52); note that t and w – and therefore also $f(t,w)$ and $\sigma(t,w)$ – are not changed in any way, and that to have a proper change of variables we must require (we recall that here Θ denotes the change of coordinates inverse to Φ) $\Phi_y \neq 0 \; \forall y$ and $\Theta_x \neq 0 \; \forall x$.

Using Ito formula,

$$\begin{aligned} dx \; &= \; \Phi_y \, dy \; + \; \Phi_t \, dt \; + \; \Phi_w \, dw \; + \; (1/2) \, \Delta(\Phi) \, dt \\ &= \; \Phi_y \, dy \; + \; \Phi_t \, dt \; + \; \Phi_w \, dw \; + \; (1/2) \left(\Phi_{yy} \, \widehat{S}^2 + \Phi_{ww} + 2 \, \widehat{S} \, \Phi_{yw} \right) dt \; ; \end{aligned} \tag{56}$$

note that here we have introduced $\widehat{S}$, i.e. the coefficient of the noise term in the equation for y, which we have not yet determined. From (56) and (53) we have

$$\Phi_y \, dy \; = \; [f - \Phi_t - (1/2)\Delta(\Phi)] \, dt \; + \; [\sigma - \Phi_w] \, dw \; ; \tag{57}$$

recalling that Φ_y is never vanishing, we divide by Φ_y and obtain an equation of the form (54) with coefficients given precisely by (55). △

Corollary 1. *The equation (54) is obtained from an equation (53) by a simple change of coordinates (33) if and only if there exist functions f, σ and Φ satisfying the equations*

$$(\sigma \; - \; \Phi_w) \; = \; \widehat{S} \, \Phi_y \; , \quad (f \; - \; \Phi_t \; - \; (1/2)\Delta\Phi) \; = \; \widehat{F} \, \Phi_y \; . \tag{58}$$

Proof. This is a trivial restatement of Lemma 2 above. △

Remark 10. Obviously eq. (53) admits $X = \partial_x$ as a Lie-point symmetry. By Lemma 1, this will also be a symmetry of the transformed equation; on the other hand, in the new coordinates we have $\partial_x = (\partial y/\partial_x)\partial_y = (1/\Phi_y)\partial_y$. Thus, in the situation considered by Lemma 2, the transformed equation (54) admits the vector field

$$X \; = \; [\Phi_y(y,t,w)]^{-1} \, \partial_y \; := \; \varphi(y,t,w) \, \partial_y \tag{59}$$

as a (simple) Lie-point symmetry, as follows at once from Lemma 1.

This can also be checked by direct computation using the explicit form of (14), (15). ⊙

We are now interested in the case where the equation (54) is actually an Ito equation,

$$\widehat{F}(y,t,w) \; = \; F(y,t) \; , \quad \widehat{S}(y,t,w) \; = \; S(y,t) \; . \tag{60}$$

Lemma 3. *Let the equation (54) be of Ito type, i.e. let (60) be satisfied. Then the functions f, σ, Φ of Lemma 2 satisfy the equations*

$$\sigma \; = \; \Phi_w \; + \; S \, \Phi_y \; ; \quad f \; = \; \Phi_t \; + \; F \, \Phi_y \; + \; (1/2) \left(\Phi_{ww} + 2 \, S \, \Phi_{yw} + S^2 \, \Phi_{yy} \right) \; . \tag{61}$$

Proof. This is exactly (58) with S taking the place of $\widehat{S}$ and writing explicitly $\Delta\Phi$. △

Lemma 4. *Let the "target" equation (54) be of Ito type, i.e. let (60) hold; moreover, let (54) be obtained from (53) by a simple random map (52). Then the "source" equation (53) is of Ito type if and only if*

$$\Phi_{ww} \; + \; S \, \Phi_{yw} \; = \; 0 \; ; \qquad \Phi_{tw} \; + \; F \, \Phi_{yw} \; + \; (1/2) \, (\Delta\Phi)_w \; = \; 0 \; . \tag{62}$$

Proof. We are in the framework of Lemma 3. The equation (53) is of Ito type if and only if the coefficients f and σ do not depend on w. The requirement that $\sigma_w = 0$ and $f_w = 0$, in view of (61), are exactly the first and second equation respectively in (62). $\triangle$

Remark 11. Introducing $\Gamma := \Phi_w$, the equations (62) are rewritten as

$$\Gamma_w \; + \; S \, \Gamma_y \; = \; 0 \; ; \quad \Gamma_t \; + \; F \, \Gamma_y \; + \; (1/2) \, (\Delta\Gamma) \; = \; 0 \; . \tag{63}$$

(To obtain these, recall that now by assumption $S_w = 0$). $\odot$

Theorem 4. *Let the Ito equation*

$$dy \; = \; F(y,t) \, dt \; + \; S(y,t) \, dw \tag{64}$$

admit as Lie-point symmetry the simple random vector field

$$X \; = \; \varphi(y,t,w) \; \partial_y \; . \tag{65}$$

If there is a determination of

$$\Phi(y,t,w) \; = \; \int \frac{1}{\varphi(y,t,w)} \; dy \tag{66}$$

such that the equations (62) are satisfied, then the equation is reduced to the explicitly integrable form

$$dx \; = \; f(t) \, dt \; + \; \sigma(t) \, dw \tag{67}$$

by passing to the variable $x = \Phi(y,t,w)$.

Proof. When we operate with the map $x = \Phi(y,t,w)$, the equation for x obtained from (64) will be written as $dx = f \, dt + \sigma \, dw$.

With such a map – for *any* determination of Φ – the vector field will X will just read as $X = \partial_x$ in the new coordinates, and we are guaranteed by Lemma 1 this will be a symmetry of the transformed equation. By (14), (15), this means that $f_x = 0$, $\sigma_x = 0$, i.e. the equation in the x coordinate will be of the form (53).

Moreover, by Lemma 4, the coefficients of this satisfy $f_w = 0 = \sigma_w$, i.e. we have a proper Ito equation, if and only if Φ satisfies the equations (62), in which now F and S are assigned functions, i.e. the coefficients of our equation (64). Thus, by choosing such a Φ (among those of the form (66)), we are guaranteed the f and σ do not depend on x nor on w, i.e. the equation is precisely of the form (67). $\triangle$

Remark 12. This Theorem is based on the existence of a determination of an integral with certain properties. It is natural to wonder how one goes in order to study the existence of such a determination. In other words, one would like to have a criterion based on the directly available data, i.e. the functions $F(y,t)$, $S(y,t)$ and $\varphi(y,t,w)$. This is provided by the next Theorem. $\odot$

Theorem 5. *Let the Ito equation (64) admit as Lie-point symmetry the simple random vector field (65); define $\gamma(y,t,w) := \partial_w(1/\varphi)$.*
If the functions $F(y,t)$, $S(y,t)$ and $\gamma(y,t,w)$ satisfy the relation

$$S \, \gamma_t \; + \; S_t \, \gamma \; = \; F \, \gamma_w \; + \; (1/2) \left[S \, \gamma_{ww} \; + \; S^2 \, \gamma_{yw} \right] \; , \tag{68}$$

then the equation (64) can be mapped into an integrable Ito equation (67) by a simple random change of variables.

Proof. For any given determination Ξ of the integral, i.e. any function satisfying $\Xi_y = (1/\varphi)$, the most general integral is given by

$$\Phi = \int \frac{1}{\varphi}\, dy = \Xi(y,t,w) - \alpha(t,w)\ ,$$

with α an arbitrary function. With this, we write

$$\Gamma = \Phi_w = \Xi_w - \alpha_w := \psi - \beta\ .$$

The equations (63) read then

$$\beta_w = \psi_w + S\,\psi_y\ , \qquad \beta_t = \psi_t + F\,\psi_y + (1/2)\,[(\Delta\psi) - \Delta(\beta)]\ . \tag{69}$$

Noting now that $\Delta\beta = \beta_{ww}$, and that in view of the first of (69) we can write $\beta_{ww} = \psi_{ww} + S\psi_{yw}$, we rewrite the equations as

$$\beta_w = \psi_w + S\,\psi_y\ , \qquad \beta_t = \psi_t + F\,\psi_y + (1/2)\left[S\,\psi_{yw} + S^2\,\psi_{yy}\right]\ . \tag{70}$$

The condition of compatibility $\beta_{tw} = \beta_{wt}$ reads therefore

$$S\,\psi_{yt} + S_t\,\psi_y = F\,\psi_{yw} + (1/2)\left[S\,\psi_{yww} + S^2\,\psi_{yyw}\right]\ . \tag{71}$$

This can be slightly simplified by writing $\gamma := \psi_y$; actually, recalling that $\psi = \Xi_w$, and that $\beta_y = 0$, this definition yields

$$\gamma(y,t,w) = \psi_y = \Xi_{wy} = \Phi_{wy} = \partial_w\,(1/\varphi)\ , \tag{72}$$

i.e. exactly the definition provided in the statement.

Thus the compatibility condition (71) can be written entirely in terms of the accessible data, i.e. the coefficients F and S in (64) and the coefficient φ of the symmetry vector field. More precisely, plugging $\gamma = \psi_y$ into (71), we get exactly (68). $\triangle$

Remark 13. Note that for φ independent of w (i.e. deterministic simple symmetries), which entails $\gamma = 0$, the equation (68) is always trivially satisfied – as it should be. $\odot$

Remark 14. Even in the case of deterministic symmetries, it can make sense to consider simple random changes of variables: by enlarging the set of considered transformations, we could obtain a simpler transformed equation (this will be the case in Example 5). $\odot$

The Theorems 4 and 5 identify the presence of a simple random symmetry (65) such that the compatibility condition (68) is satisfied as a *sufficient* condition for integrability. It is quite simple to observe this is also a *necessary* condition.

Theorem 6. *Let the Ito equation (64) be reducible to the integrable form (67) by a simple random change of variables (52). Then necessarily (64) admits (59) as a symmetry vector field, and – with $\gamma = \partial_w(1/\varphi)$ – (68) is satisfied.*

Proof. Let $y = \Theta(x,t,w)$ be the change of variables inverse to (52). By assumption, there is an integrable Ito equation (67) which is mapped into (64) by Θ. Equation (67) admits $X = \partial_x$ as a symmetry, and by Lemma 1 this will also be a symmetry of (64); on the other hand, X is written in the (y,t,w) coordinates exactly in the form (59). The fact that (68) is satisfied follows at once from the assumption that both (67) and (64) are Ito equations. $\triangle$

Thus, in the end, even within the framework of (simple) random maps and (simple) random symmetries, the presence of a simple random symmetries – now with the additional requirement that a compatibility condition (68) is satisfied – is a *necessary and sufficient* condition for a given (scalar) Ito equation to be integrable by a (simple random) change of variables.

VIII. EXAMPLES OF INTEGRATION VIA RANDOM SYMMETRIES

In this Section we will present several examples of reduction to integrable form via (changes of variables identified by) simple random symmetries. As this kind of reduction is new in the literature, we will discuss these examples in greater detail than the previous ones.

Example 5. Let us consider the equation

$$dy = \left((1/2)\, t^2\, e^{-t} - y\right) dt + (t+k)\, e^{-t}\, dw ; \qquad (73)$$

this admits as symmetry $X = e^{-t}\partial_y$. Here $\varphi_w = 0$ and the equation (68) is trivially satisfied. According to our general result, we should therefore use the change of variable described by

$$\Phi = \int e^t\, dy = e^t\, y + \beta(t,w) ; \qquad (74)$$

with this, the equation (73) is mapped into

$$dx = \left(\frac{t^2}{2} + (1/2)\,\beta_{ww} + \beta_t\right) dt + (k+t+\beta_w)\, dw . \qquad (75)$$

As expected, its coefficients are always independent of y.

For this to be an Ito equation, from the coefficient of dw it is needed to have $\beta_{ww} = 0$, hence $\beta(t,w) = b(t)+b^{(1)}(t)w$, and from the coefficient of dt we get that actually we should require $b^{(1)}(t) = c$. In other words, we have

$$\Phi = c\,w + e^t\, y + b(t) , \qquad (76)$$

where the constant c and the function $b(t)$ are arbitrary. With the map $x = \Phi(y,t,w)$ and this choice of Φ, we get the equation

$$dx = \left(\frac{t^2}{2} + b'(t)\right) dt + (k+c+t)\, dw := f(t)\, dt + \sigma(t)\, dw; \qquad (77)$$

this is always (that is, for any choice of $b(t)$ and c) integrable.

We can get a somewhat simpler equation choosing $c = -k$ and $b(t) = 0$, which corresponds to $\Phi(y,t,w) = -kw+e^t y$; now the transformed equation is[?]

$$dx = \frac{t^2}{2}\, dt + t\, dw . \qquad (78)$$

Note that in this case the symmetry is actually deterministic; if we were working in the framework of deterministic simple maps, the only difference would have been that we should have just considered $c = 0$ and $\beta(t,w) = b(t)$. ⊙

Example 6. The equation

$$dy = -\left(e^{-y} + (1/2)\, e^{-2y}\right) dt + e^{-y}\, dw \qquad (79)$$

admits the vector field

$$X = \frac{e^{-y}}{t + e^y - w + 1}\, \partial_y := \varphi(y,t,w)\, \partial_y \qquad (80)$$

as symmetry (this is a genuine random symmetry); condition (68) is satisfied. We hence have

$$\Phi = \int \frac{1}{\varphi(y,t,w)}\, dy = e^y(t - w + 1) + \frac{e^{2y}}{2} + \beta(t,w) . \qquad (81)$$

By passing to the variable $x = \Phi(y,t,w)$ the equation (79) reads

$$dx = \left(\beta_t + (1/2)\,\beta_{ww} + w - t - \frac{3}{2}\right) dt + (\beta_w - w + t + 1)\, dw := f\, dt + \sigma\, dw .$$

Again the coefficients are – as expected – always independent of y.

In order to have $\sigma_w = 0$ we must require $\beta_{ww} = 1$, i.e. $\beta(t,w) = b(t) + [b^{(1)}(t)]w + (1/2)w^2$; with this, requiring $f_w = 0$ enforces $b^{(1)}(t) = c_1 - t$. The equation reads now

$$dx = -(1 + t + b'(t))\,dt + (1 + c_1)\,dw , \tag{82}$$

which is obviously integrable. Note that with our choice for β we have got

$$\Phi(y,t,w) = \frac{w^2}{2} + (c_1 - e^y - t)\,w + \left[e^y + \frac{e^{2y}}{2} + e^y\,t + b(t)\right] . \tag{83}$$

By choosing $c_1 = 0$, $b(t) = 0$ we get slightly simpler expressions: the transformed equation is

$$dx = -(1 + t)\,dt + dw , \tag{84}$$

and the integrating map is

$$\Phi(y,t,w) = \frac{w^2}{2} - (e^y + t)\,w + \left[1 + \frac{e^y}{2} + t\right]e^y . \tag{85}$$

Note that (79) also admits a simple *deterministic* symmetry, i.e. $X_0 = e^{-y}\partial_y$.[?] $\odot$

Example 7. Consider the Ito equations

$$dy = y\,p(t)\,dt + y\,dw , \tag{86}$$

for $p(u)$ any smooth function. The determining equation (15) reads in this case $\varphi = \varphi_w + y\varphi_y$ and hence yields $\varphi(y,t,w) = y\psi(z,t)$, where $z := ye^{-w}$; with this the other determining equation (14) reads $\psi_t + z[p(t) - (1/2)]\psi_z = 0$ and yields $\psi(z,t) = \eta(\zeta)$, where we have defined

$$\zeta = z\,\exp[R(t)] = y\,\exp[-w + R(t)] ; \quad R(t) := \int_0^t \left(\frac{1}{2} - p(u)\right)du . \tag{87}$$

Thus, in the end, the simple symmetries of (86) are identified by

$$\varphi(y,t,w) = y\,\eta(\zeta) , \tag{88}$$

with η an arbitrary non-zero function. The compatibility condition (68) is always satisfied.

The change of variables identified by such symmetries are

$$x = \Phi(y,t,w) = \int \frac{1}{y\,\eta(\zeta)}\,dy + \beta(t,w) . \tag{89}$$

In order to apply Lemma 3, and in particular (61), we compute partial derivatives of Φ; in doing this we should distinguish the case $\eta = K$ (with $K \neq 0$ a constant) and the generic one.

In fact, if $\eta'(\zeta) \neq 0$, we have

$$\Phi_t = \beta_t - R'(t)\,e^{-w+R(t)}\int(\eta_\zeta/\eta^2)\,dy = \beta_t - R'(t)\int(\eta_\zeta/\eta^2)\,d\zeta = \beta_t + \frac{R'(t)}{\eta} ,$$

$$\Phi_w = \beta_w + e^{-w+R(t)}\int(\eta_\zeta/\eta^2)\,dy = \beta_w + \int(\eta_\zeta/\eta^2)\,d\zeta = \beta_w - \frac{1}{\eta} ;$$

by the same computation it is clear that in the case $\eta = K$, and hence $\eta_\zeta = 0$, we get

$$\Phi_t = \beta_t , \quad \Phi_w = \beta_w .$$

Obviously we always have $\Phi_y = 1/(y\eta)$.

As for the second order partial derivatives, these are (these formulas also hold for $\eta = K$, simply setting $\eta_\zeta = 0$ in them)

$$\Phi_{yy} = -\frac{\eta + \zeta\,\eta_\zeta}{y^2\,\eta^2} , \quad \Phi_{ww} = \beta_{ww} - \frac{\zeta\,\eta_\zeta}{\eta^2} , \quad \Phi_{wy} = \frac{\zeta\,\eta_\zeta}{y\,\eta^2} ;$$

With these – and recalling $S = y$, $F = yp(t)$, see (86) – we always get $\Delta(\Phi) = \beta_{ww} - (1/\eta)$. We can now apply (61) to get f and σ of the transformed equation.

(A) *(η not constant)*. For $\eta \neq K$, applying the first of (61) we get immediately

$$\sigma = \beta_w . \tag{90}$$

Applying the second of (61), for η non constant we get $f = [-1 + 2p(t) + 2R'(t) + \eta\beta_{ww} + 2\eta\beta_t]/(2\eta)$; recalling now the expression for $R(t)$, hence $R'(t) = (1/2) - p(t)$, this just reduces to

$$f = \beta_t + \frac{1}{2}\beta_{ww} . \tag{91}$$

If now we require that f and σ are both independent of w, we get

$$\beta(t, w) = b(t) + cw .$$

With this – recalling (90) and (91) – we obtain[**?**]

$$dx = [b'(t)] dt + c dw . \tag{92}$$

This shows that we can have several random symmetries – actually, in this case, infinitely many ones, depending on the choice of an arbitrary non-constant smooth function η – leading to *the same* integrable equation; this situation is discussed in more detail in B.

(B) *(η constant)*. For η constant, applying the first of (61) we get

$$\sigma = \beta_w + K^{-1} ; \tag{93}$$

applying now also the second of (61) we get

$$f = \left(b_t + \frac{1}{2}\beta_{ww}\right) + \left(p(t) - \frac{1}{2}\right) K^{-1} . \tag{94}$$

In order to have $s_w = 0$ and $f_w = 0$ we must again require $\beta = b(t) + cw$. Thus for $\eta = K$ the reduced equation is

$$dx = \frac{1}{K} \left[\left(p(t) - \frac{1}{2} + K b'(t)\right) dt + (1 + K c) dw\right] . \tag{95}$$

Note also that if we choose η to be constant, we have a *deterministic* symmetry; this leads to a *different* reduction (for the same choice of $\beta(t, w)$; it is clear that by suitable different choices of β the two reductions can be made equal), as seen by comparing (92) and (95). ⊙

Example 8. Consider the Ito equation

$$dy = dt + y dw ; \tag{96}$$

the only simple Lie-point symmetry this admits is

$$X = e^{w-t/2} \partial_y := \varphi(y, t, w) \partial_y . \tag{97}$$

In fact, in this case eq.(15) yields immediately

$$\varphi(y, t, z) = y \psi(z, t), \qquad z = ye^{-w} ;$$

plugging this into (14) we obtain

$$(\psi + z \psi_z) + (\psi_t - (1/2) z \psi_z) y = 0 .$$

Each of the two terms in brackets must vanish, hence $\psi(z, t) = \eta(t)/z$ and $\eta = ke^{-t/2}$. Going back to the original variables and function, we obtain (97).

For this form of φ, we have $\gamma = \partial_w(1/\varphi) = e^{t/2-w}$, and it is easy to check that the compatibility condition (68) is *not* satisfied.

Let us now consider the changes of variables (52) such that $X = \partial_x$ in the new variables, i.e. yielding equations of the form (53); we know these are of the form

$$\Phi(y, t, w) \;=\; \int \frac{1}{\varphi} dy \;=\; \left(e^{t/2-w}\right) y \;+\; \beta(t, w) \;.$$

The coefficients f, σ in (53) are given in this case by (61), and we obtain

$$\sigma \;=\; \beta_w \;, \quad f \;=\; e^{t/2-w} \;+\; \frac{1}{2} \beta_{ww} \;+\; \beta_t \;; \tag{98}$$

requiring $\sigma_w = 0$ would enforce $\beta_{ww} = 0$, i.e.

$$\beta(t, w) \;=\; b_0(t) \;+\; b_1(t)\, w \;;$$

with this, we get $f = e^{t/2-w} + \beta_0' + w b_1'$, and the requirement $f_w = 0$ amounts to

$$b_1'(t) \;=\; e^{t/2-w} \;,$$

which does not admit any solution.

Thus in this case the equation admits a simple random symmetry, but it can *not* be transformed into an integrable Ito equation by a simple random map; this corresponds to the fact that the compatibility condition (68) is *not* satisfied. $\odot$

IX.   CONCLUSIONS AND DISCUSSION

We have considered in constructive terms the relation between *symmetry* and *integrability* – or at least *reducibility* – for stochastic differential equations (or systems thereof).

We started by recalling that symmetries of an Ito equation are defined only algebraically, not geometrically; thus – contrary to the deterministic case or even to the Stratonovich one – in this context it is not granted that symmetry are preserved under a change of variables. On the other hand, preservation of *simple deterministic* symmetries (which are exactly the symmetries relevant in Kozlov theory [18–20]) of Ito equations is guaranteed, as can be shown by considering the relation with symmetries of the associated Stratonovich equations [13, 32].

We have then considered Kozlov results [18–20]; our Theorems 1, 2 and 3 reproduce them, but while Kozlov was interested in providing *sufficient* symmetry conditions for integrability or reducibility, we have also considered *necessary* ones; actually showing that Kozlov conditions are *necessary and sufficient*. Moreover, we have provided fully explicit and complete proofs to his results.

On the other hand, one can consider more general sets of maps and hence symmetries; in particular, we have considered *simple random maps and symmetries* [15]. In this case the preservation of symmetries of an Ito equation is granted by a recent result of ours [13].

We have then tackled, confining our study to the scalar case, the problem of extending Kozlov theory to the framework of (simple) random maps and symmetries. This extension is provided by Theorem 5 and Theorem 6, which are a direct extension of Kozlov theorem [18] and of our completion of it (Theorem 1), except that now we have to ask further conditions (62) to ensure we obtain an Ito equation, or equivalently a compatibility condition (68) to be sure they can be satisfied.

Some final considerations are in order here, also regarding possible further extensions of Kozlov theory.

(a) Our study of random maps and possibly random symmetries was limited to scalar equations; we trust it can be extended to the case of systems, but such study is deferred to a forthcoming contribution.

(b) A "naive" extension of Kozlov theory would consider reduction to SDEs which can be integrated albeit not being of the form considered here (and hence not "immediately" integrable); this would be e.g. the case for separable SDEs. As mentioned above, in this case one does not obtain anything new w.r.t. the standard Kozlov theory; this is discussed in detail in A.

(c) Further generalizations can be envisaged, albeit they have not been considered here. The discussion by Kozlov [18–20] suggests that there is no point in considering deterministic symmetries which are not simple, i.e. which also act on t; on the other hand, there is probably a point in considering W-symmetries [11], which will be done elsewhere.

(d) Also, here we have mainly investigated the relation between symmetries and integrability, which corresponds to an extremely simplified form of SDE; but other – less radical – simplification of the SDEs under study can also be

of interest. One has been considered in Section V, where we considered systems which are reducible to a system of lower dimension, albeit not integrable, plus some "reconstruction equations". One could also consider e.g. systems which are separable into two or more subsystems, with no "reconstruction equation". Such systems can also be characterized, albeit less immediately, in symmetry terms and it is thus possible that systems which can be brought into such form can also be detected in terms of their symmetry properties.

(e) In this note we have only considered Ito equations. One could of course consider also more general stochastic differential equations and their symmetries (see e.g. the recent paper [1]; some of its authors also propose an alternative approach to reduction and reconstruction of stochastic differential equations via symmetries [8]). In this framework the approach presented here can be of use to identify such more general equations which can be mapped back to an Ito integrable equation.

(f) Finally we note that, as in the deterministic case, one expects that the relation between symmetries and integrability displays special features in the case of stochastic equations with a *variational* origin. This lies beyond the limit of the present study but several results exist in this direction, see e.g. [2, 21, 31, 33, 34].

APPENDIX A: THE "EXTENDED" KOZLOV THEOREM

Equations of the form (17) are not the only SDEs which can be integrated. Another class is provided by equations of the form

$$dx = \beta(x)\, f(t)\, dt + \beta(x)\, \sigma(t)\, dw . \qquad (A.1)$$

Note that these are integrated to provide the solution in implicit form only (in general), i.e. as

$$B(x) := \int \frac{1}{\beta(x)}\, dx = \int f(t)\, dt + \int \sigma(t)\, dw .$$

Correspondingly we would have an extension of Theorem 1 to consider this case. It turns out, however, that equations of the form (A.1) do not in general admit simple Lie symmetries. This is detailed in Lemma A.1 below.

Lemma A.1. *The equation (A.1) does not admit any simple Lie point symmetry generator unless $\beta(x) = x$; in this case the admitted symmetry generator is*

$$X = x\, \partial_x . \qquad (A.2)$$

Proof. The equation (9) reads in this case

$$\sigma\, (\beta\, \varphi_x - \varphi\, \beta_x) = 0 ,$$

and hence (for $\sigma \neq 0$) has solution

$$\varphi(x,t) = A(t)\, \beta(x) .$$

Plugging this general expression for φ into (8), we get

$$A'(t)\, \beta(x) = -(1/2)\, [\beta(x)\, \sigma(t)]^2\, A(t)\, \beta_{xx}(x) .$$

This in turn implies

$$\frac{A'}{A\,\sigma} \; = \; -(1/2)\beta\beta_{xx} \; ;$$

but here the expression on the l.h.s. is a function of t alone, that on the r.h.s. of x alone. Hence we must have

$$\frac{A'}{A\,\sigma} \; = \; c_0 \; = \; -(1/2)\beta\beta_{xx} \; ,$$

with c a numerical constant.

The first of these equations yields

$$A(t) \; = \; c_1 \; \exp\left[c_0 \; \int g^2(t) \; dt \right]$$

with c_1 an arbitrary constant; the second one requires

$$\beta \; \beta_{xx} \; = \; -2c_0 \; ,$$

and the solution is given by

$$\beta(x) \; = \; b_0 \; x \; , \tag{A.3}$$

with b_0 another constant. This also implies $c_0 = 0$, and hence $A(t) = c_1$.

In other words, (A.1) has a *simple* Lie symmetry[**?**] if and only if $\beta(x)$ is of the form (A.3) (then we can move the constant b_0 into the $f(t)$ and $g(t)$ functions, i.e. we can assume $b_0 = 1$ with no loss of generality). $\triangle$

It should be noted that for $\beta(x)$ of the form (A.3) and the associated (A.1), the determining equations give (A.2). The symmetry group generated by (A.2) is just a scaling one, $x \to \lambda x$.

As mentioned above, one could in principles follow the approach of Theorem 1 and get a result similar to the one there for equations which can be reduced to the form (A.1). However, we have seen that a characterization in terms of simple symmetries is possible only for β as in (A.3). But for such a case we can come back to the case considered by Kozlov.

Lemma A.2. *The equation (A.1) in the case (A.3) is mapped to an equation of the form (17) by the change of variables*

$$x \; = \; \Phi(y) \; := \; e^y \; .$$

Proof. By direct computation. $\triangle$

APPENDIX B: MULTIPLE SIMPLE RANDOM SYMMETRIES

We have observed, in Examples 7 and 8, that one can have a situation where random symmetries correspond to a certain pattern in which enters an arbitrary function. Here we want to discuss this situation – and the conditions in which it can occur – in some detail. We will again confine ourselves to scalar Ito equations, as in Sections VII and VIII.

First of all, given an Ito equation[**?**] (1) we introduce the linear operators

$$L \; := \; \partial_t \; + \; \left[f(x,t) \; - \; \frac{1}{2}\,\sigma(x,t)\,\sigma_x(x,t) \right] \partial_x \; ; \quad M \; := \; \partial_w \; + \; \sigma(x,t)\,\partial_x \; . \tag{B1}$$

With these, the determining equations (14), (15) are written as

$$L(\varphi) \; + \; \frac{1}{2}\,M^2(\varphi) \; = \; \varphi\,(\partial_x F) \; ,$$
$$M(\varphi) \; = \; \varphi\,(\partial_x \sigma) \; . \tag{B2}$$

Suppose now that we have a solution $\chi(x,t,w)$ to these equations, and let us look for solutions in the form $\varphi = \psi\chi$. Plugging this into (B2), we obtain

$$\chi L(\psi) + \psi L(\chi) + \frac{1}{2}\left[\chi M^2(\psi) + 2M(\psi)M(\chi) + \psi M^2(\chi)\right] = \psi\chi\,(\partial_x F)\,,$$
$$\chi M(\psi) + \psi M(\chi) = \psi\chi\,(\partial_x\sigma)\,. \tag{B3}$$

Recalling now that χ is, by assumption, a solution to (B2), these reduce to

$$\chi L(\psi) + \frac{1}{2}\left[2M(\chi)M(\psi) + \chi M^2(\psi)\right] = 0\,,\quad \chi M(\psi) = 0\,. \tag{B4}$$

The second equation requires $\psi \in \mathrm{Ker}(M)$, and assuming this the first ones reduces to $\psi \in \mathrm{Ker}(L)$. Thus ψ is any function in

$$\mathcal{F} = \mathrm{Ker}(L) \cap \mathrm{Ker}(M)\,. \tag{B5}$$

Looking back at *Example 7*, in that case $f = yp(t)$, $\sigma = y$; hence (with the present notation) $L = \partial_t + [p(t) - 1/2]x\partial_x$, $M = \partial_w + x\partial_x$. It is then immediate to obtain that $\mathrm{Ker}(L)$ consists of arbitrary functions $P[z_\ell, w]$, while $\mathrm{Ker}(M)$ consists of arbitrary functions $Q[z_m, t]$ where

$$z_\ell = x\,\exp\left[-\int_0^t [p(u) - 1/2]\,du\right]\,,\quad z_m = x\,e^{-w}\,.$$

In this case, $\mathcal{F} = \mathrm{Ker}(L) \cap \mathrm{Ker}(M)$ is non-trivial, and consists of arbitrary functions

$$\eta\left(x\,\exp\left[-w - \int_0^t [p(u) - 1/2]\,du\right]\right)\,. \tag{B6}$$

To be more concrete, e.g. with the choice $p(t) = 1$ we get $\mathcal{F} = \eta\left(x\,e^{-w-t/2}\right)$, while for $p(t) = t$ we get $\mathcal{F} = \eta\left(x\,e^{-w+t/2-t^2/2}\right)$.

Let us now look at *Example 8*. In this case $f = 1$, $\sigma = y$; hence (with the present notation) $L = \partial_t + (1 - x/2)\partial_x$, $M = \partial_w + x\partial_x$. In this case we get that again $\mathrm{Ker}(L)$ and $\mathrm{Ker}(M)$ consist respectively of arbitrary functions $P[z_\ell, w]$ and $Q[z_m, t]$, where now

$$z_\ell = (x - 2)\,e^{t/2}\,,\quad z_m = x\,e^{-w}\,.$$

With these, $\mathcal{F}$ is trivial, i.e. reduces to constant functions.

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[1] S.Albeverio, F. De Vecchi, P. Morando and S. Ugolini, "Symmetries and invariance properties of stochastic differential equations driven by semimartingales with jumps", preprint *arXiv:1708.01764*

[2] S. Albeverio, J. Rezende and J.C. Zambrini, "Probability and quantum symmetries. II. The theorem of Noether in quantum mechanics", *J. Math. Phys.* **47** (2006), 062107

[3] D.V. Alexseevsky, A.M. Vinogradov and V.V. Lychagin, *Basic Ideas and Concepts of Differential Geometry*, Springer 1991

[4] L. Arnold, *Random dynamical systems*, Springer 1988

[5] L. Arnold and P. Imkeller, "Normal forms for stochastic differential equations", *Prob. Th. Rel. Fields* **110** (1998), 559-588

[6] S.S. Chern, W.H. Chen and K.S. Lam, *Lectures on Differential Geometry*, World Scientific, 2000

[7] G. Cicogna and G. Gaeta, *Symmetry and perturbation theory in nonlinear dynamics*, Springer 1999

[8] F. De Vecchi, P. Morando and S. Ugolini, "Reduction and reconstruction of stochastic differential equations via symmetries", *J. Math. Phys.* **57** (2016), 123508

[9] L.C. Evans, *An introduction to stochastic differential equations*, A.M.S. 2013

[10] D. Freedman, *Brownian motion and diffusion*, Springer 1983

[11] G. Gaeta, "Lie-point symmetries and stochastic differential equations II", *J. Phys. A* **33** (2000), 4883-4902

[12] G. Gaeta, "Symmetry of stochastic non-variational differential equations", *Phys. Rep.* **686** (2017), 1-62 [Erratum: *Phys. Rep.* **713** (2017), 18]

[13] G. Gaeta and C. Lunini, "On Symmetries of stochastic differential equations", *J. Nonlin. Math. Phys.* **24-S1** (2017) 90-102

[14] G. Gaeta and N. Rodríguez-Quintero, "Lie-point symmetries and stochastic differential equations", *J. Phys. A* **32** (1999), 8485-8505

[15] G. Gaeta and F. Spadaro, "Random Lie-point symmetries of stochastic differential equations", *J. Math. Phys.* **58** (2017), 053503 [Erratum: to appear in *J. Math. Phys.*]

[16] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, North Holland 1981

[17] N.G. van Kampen, *Stochastic processes in Physics and Chemistry*, North Holland 1992

[18] R. Kozlov, "Symmetry of systems of stochastic differential equations with diffusion matrices of full rank", *J. Phys. A* **43** (2010), 245201

[19] R. Kozlov, "The group classification of a scalar stochastic differential equations", *J. Phys. A* **43** (2010), 055202

[20] R. Kozlov, "On maximal Lie point symmetry groups admitted by scalar stochastic differential equations", *J. Phys. A* **44** (2011), 205202

[21] T. Misawa, "Noether's theorem in symmetric stochastic calculus of variations", *J. Math. Phys.* **29** (1988), 2178-2180

[22] M. Nakahara, *Geometry, Topology and Physics*, IOP, 1990

[23] Ch. Nash and S. Sen, *Topology and Geometry for Physicists*, Academic Press, 1983; Dover, 2011

[24] B. Oksendal, *Stochastic differential equations* (4th edition), Springer 1985

[25] P.J. Olver, *Application of Lie groups to differential equations*, Springer 1986

[26] P.J. Olver, *Equivalence, Invariants and Symmetry*, Cambridge University Press 1995

[27] L.S. Schulman, *Techniques and applications of path integration*, Wiley 1981; reprinted by Dover 2005

[28] B. Srihirun, S.V. Meleshko and E. Schulz, "On the definition of an admitted Lie group for stochastic differential equations with multi-Brownian motion", *J. Phys. A* **39** (2006), 13951-13966

[29] H. Stephani, *Differential equations. Their solution using symmetries*, Cambridge University Press 1989

[30] D.W. Stroock, *Markov processes from K.Ito's perspective*, Princeton UP 2003

[31] M. Thieullen and J.C. Zambrini, "Probability and quantum symmetries. I. The theorem of Noether in Schrodinger's euclidean quantum mechanics", *Ann. IHP Phys. Théor.* **67** (1997), 297-338

[32] G. Unal, "Symmetries of Ito and Stratonovich dynamical systems and their conserved quantities", *Nonlin. Dyn.* **32** (2003), 417-426

[33] K. Yasue, "Stochastic calculus of variations", *Lett. Math. Phys.* **4** (1980), 357-360; *J. Funct. Anal.* **41** (1981), 327-340

[34] J.C. Zambrini, "Stochastic dynamics: A review of stochastic calculus of variations", *Int. J. Theor. Phys.* **24** (1985), 277-327

[35] By "smooth" we will always mean C^∞, albeit in most steps it would be sufficient to consider C^2 smoothness.

[36] This and the following example are also given in [13]; see there for details.

[37] An even more radical simplification is obtained by choosing $c = -k$ and $b(t) = -t^3/6$, i.e. $\Phi = -kw + e^t y - t^3/6$; with this choice the transformed equation is $dx = t dw$.

[38] More generally, any function $\varphi(y, t, w) = e^{-y}\eta(e^y + t - w)$ identifies a Lie-point symmetry for (79); see also the following Example 7 and Appendix B in this respect.

[39] Note that in this case we obtain $dx = d\beta$; indeed it follows directly from eq.(61) that choosing Φ to be just the integral part of (89) (i.e. setting $b(t, w) = 0$) we get $f(x, t) = \sigma(x, t) = 0$. Note also this is not the case in Example 6.

[40] Equations with different $\beta(x)$ could in principles admit more general symmetry generators, i.e. could have symmetries which are not simple ones. We will not discuss these here.

[41] We stress this is written in terms of the x variable, and correspondingly the coefficient of the Wiener process in the equation is $\sigma(x, t)$: in this framework – and hence in this Appendix – the Ito Laplacian is written exactly as in (10).