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HIGHER ORDER NORMAL MODES

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Dedicated to James Montaldi on his 25+something anniversary

Abstract. Normal modes are intimately related to the quadratic approximation of a potential at its hyperbolic equilibria. Here we extend the notion to the case where the Taylor expansion for the potential at a critical point starts with higher order terms, and show that such an extension shares some of the properties of standard normal modes. Some symmetric examples are considered in detail.

1. **Introduction.** The concept of *normal modes* is a fundamental one in the study of Hamiltonian dynamical systems [1,11,13,15]; it is based on the quadratic part of the Taylor expansion of the Hamiltonian around a non-degenerate (isolated and hyperbolic) stable equilibrium, and under certain fairly general assumptions it can be conveniently employed also in considering higher order expansions of the Hamiltonian. That is, under such assumptions (including a non-resonance condition) normal modes persist, at least locally, when one considers also higher order terms – or for nonlinear Hamiltonian dynamics [17, 29] – as also studied by James in a series of papers [18,19].

Normal modes span the dynamics of the quadratic Hamiltonian, i.e. any motion for this can be described as the superposition of normal modes; more precisely, normal modes define invariant lines in a neighborhood $U \simeq R^n$ of the equilibrium in the position space, and these provide a basis for U.

Here we want to discuss the (partial) generalization of this concept to the case where the equilibrium is stable and isolated, but *not* hyperbolic. This resonates with some recent studies in a different field, i.e. *liquid crystals*. More precisely, we will find some connection with recent studies of liquid crystals described by higher order tensor order parameter [8,10,12,28].

We will show that normal modes – in the sense of invariant lines – also exist for the dynamics of fully nonlinear homogeneous Hamiltonian systems (more precisely, we will confine ourselves to systems with a natural Hamiltonian $H = T + V$ and

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a potential $V(x)$ homogeneous of degree $k > 2$). On the other hand, unless some very special conditions (e.g. based on the symmetry of the full system) are met, it is *not* possible to describe a generic motion as a (nonlinear) superposition of these higher order normal modes.

2. **Normal modes.** We will consider natural Hamiltonians

$$H \;=\; T \;+\; V \;=\; \frac{\mathbf{p}^2}{2\,m} \;+\; V(\mathbf{q}) \tag{1}$$

with $\mathbf{q} \in \mathbf{R}^n$; the potential V is such that the origin is an isolated stable equilibrium point.

Our discussion could be extended to encompass a general Hamiltonian on a symplectic manifold M of dimension $2n$, and a non-degenerate equilibrium point p_0 for this; however our considerations will be *local*, so by Darboux' theorem we can always consider a standard symplectic form.

The physical interest of the natural Hamiltonian case surely justifies considering this special case. Moreover this will help keeping the discussion and notation simpler, focusing on the key points.[1]

We can then consider the Taylor expansion of V around $\mathbf{q} = 0$; we thus have, in view of the non-degeneracy assumption,

$$V(\mathbf{q}) \;=\; V_0 \;+\; \frac{1}{2}\,(\mathbf{q}, A\mathbf{q}) \;+\; \texttt{h.o.t.}\,,$$

where $V_0 = V(\mathbf{0})$ is a constant – which can be set to zero with no loss of generality – and A is a symmetric tensor of order two, i.e. a symmetric matrix,

$$A_{ij} \;=\; \left(\frac{\partial^2 V}{\partial q^i \partial q^j}\right)_0 \,. \tag{2}$$

We will now truncate the potential at order two, i.e. omit the higher order terms and deal just with $V = 1/2(\mathbf{q}, A\mathbf{q})$; the relevant case for normal modes is that where the origin is a stable fixed point, hence A is a positive definite matrix.

This matrix A will have eigenvalues λ_i and corresponding (say normalized) eigenvectors $\phi_{(i)}$; these in turn correspond to normal modes in a very well known way, which we briefly recall to fix notation [1, 11, 13, 15].

1. The eigenvectors $\phi_{(i)}$ identify *invariant lines*: if a motion has initial conditions $\mathbf{q}(0) = \alpha(0)\phi_{(i)}$, $\dot{\mathbf{q}}(0) = \beta(0)\phi_{(i)}$, then $\mathbf{q}(t) = \alpha(t)\phi_{(i)}$ (and hence $\dot{\mathbf{q}}(t) = \beta(t)\phi_{(i)}$) for all t.
2. The eigenvectors $\phi_{(i)}$ $(i = 1, ..., n)$ span the linear space $\mathbf{R}^n$;
3. A general motion, i.e. one with generic initial conditions $\mathbf{q}(0)$, $\dot{\mathbf{q}}(0)$ can then be decomposed in terms of normal modes. In fact, by the property just mentioned, there will exist constants $\alpha_i(0)$, $\beta_i(0)$ such that

$$\mathbf{q}(0) \;=\; \sum_{i=1}^{n} \alpha_i(0)\,\phi_{(i)}\;; \quad \dot{\mathbf{q}}(0) \;=\; \sum_{i=1}^{n} \beta_i(0)\phi_{(i)}\,.$$

The evolution of the system will then just be described by

$$\mathbf{q}(t) \;=\; \sum_{i=1}^{n} \alpha_i(t)\,\phi_{(i)}\;; \quad \dot{\mathbf{q}}(t) \;=\; \sum_{i=1}^{n} \beta_i(t)\phi_{(i)}\,,$$

[1] In particular, all the discussion in this section, i.e. referring to the quadratic case, can easily be set in terms of a general Hamiltonian $H(\mathbf{p}, \mathbf{q})$.

with $\alpha_i(t)$ and $\beta_i(t)$ being exactly as in the normal mode solution with initial data $\{\alpha_i(0), \beta_i(0)\}$. In other words, any solution for this quadratic Hamiltonian will be a *linear superposition of normal modes solutions.*

4. As a consequence of the previous item, in the case there exist several normal modes associated with the same frequency, the whole linear space spanned by the associated eigenvectors is made of eigenvectors and hence of normal modes with the given frequencies.

As anticipated, we will find that properties (1) and (2) extend to the higher order case, while properties (3) and (4), related to linearity, do not.

3. **Eigenvectors of tensors.** We now consider the case where $\mathbf{q} = 0$ is an isolated critical point, but we will drop the hypothesis it is hyperbolic. Actually, we want to assume that the matrix A defined in (2) is *identically zero*, i.e. that the potential V is *fully nonlinear* at the considered critical point.

Thus we have to go further in the Taylor expansion – even if we want to stop at the first significant term – and actually if we want the critical point to be a stable one it is needed to consider a fourth order[2] tensor T,

$$T_{ijk\ell} \;:=\; \frac{1}{24}\left(\frac{\partial^4 H}{\partial q^i \partial q^j \partial q^k \partial q^\ell}\right)_0 . \tag{3}$$

Thus, disregarding higher order terms, we want to consider a potential expressed in local coordinates which is homogeneous of order four and which has an isolated equilibrium at the origin. After getting rid of inessential (additive and multiplicative) constants, this can be written as

$$V \;=\; T_{ijk\ell}\; q^i\, q^j\, q^k\, q^\ell \; . \tag{4}$$

The "direct extension" of properties (1) and (2) recalled above for usual normal modes would require to consider *eigenvectors for the fourth order tensor* T (rather than the matrix A) and see if these span the whole space.

The notion of eigenvector of tensors is not so well known in general, but it can be defined and it has been studied (with a revival of interest in recent times) both from the algebraic point of view [5,20,21] and in connection with dynamics [14,22–26,30] (see also [4, 31]); as already mentioned, it has also been recently considered in connection with critical points of a constrained potential with applications in the Physics of Liquid Crystals [8,10,12,28].

We will not introduce and discuss the notion of eigenvectors of tensors right away, but we will start with a general discussion of one dimensional eigenspaces of homogeneous polynomial maps.

3.1. **Eigenspaces of homogeneous polynomial vector fields.** We will simultaneously discuss the real and the complex case in this section. We first recall a classical result from Algebraic Geometry; see [27] for more about this.

Theorem 1 (Bezout). *Let* $\{f_1, \dots f_m\}$ *be homogeneous polynomials of degree* n, $f_i : \mathbf{C}^m \to \mathbf{C}$. *Then the number of common zeros of the* f_i *in projective space* $\mathbf{P}^m$ *is either infinite or equal to* n^m, *counting multiplicities.*

For real polynomials this theorem provides only limited information about the nature of these critical points; in particular, we cannot infer how many of these are

[2]This is quite different from the case of liquid crystals mentioned above: there the relevant tensor is of order three [8,10,12,28].

real.[3] But, since the complex conjugate of every solution is also a solution (with the same multiplicity, as can be shown), we conclude that a real solution exists whenever n is odd (and there are only finitely many solutions).

We now denote by $\mathbf{K}$ the real or complex numbers and consider a homogeneous polynomial map

$$B : \mathbf{K}^q \to \mathbf{K}^q, \quad x \mapsto \begin{pmatrix} B^1(x) \\ \vdots \\ B^q(x) \end{pmatrix} \tag{5}$$

with each B^i homogeneous of degree $p \geq 2$. In coordinates we have

$$B^i = B^i_{j_1 \dots j_q} x^{j_1} \dots x^{j_q}. \tag{6}$$

We stipulate that the coefficients $B^i_{j_1 \dots j_q}$ are symmetric with respect to permutations of $j_1, \dots, j_q$; this choice makes the coefficients unique. It is possible to identify B with its coefficients, and consider it as an element of the coefficient space (which is just some $\mathbf{K}^N$.) A formal definition follows next.

Definition. *Let B be given as in (5). Then a nonzero $v \in \mathbf{K}^q$ is called an eigenvector of B if there exists an $\alpha \in \mathbf{K}$ such that $B(v) = \alpha v$.*

Some remarks are in order here:

1. Every nonzero scalar multiple of an eigenvector v is also an eigenvector. Therefore it makes sense to call $\mathbf{K}\, v$ an eigenspace of B.
2. On the other hand, the notion of eigenvalue is problematic for homogeneous maps of degree > 1, since
$$B(\beta v) = \beta^p \alpha v = (\beta^{p-1}\alpha) \cdot (\beta v)$$
for any β, and thus one may replace the "eigenvalue" α by $\beta^{p-1}\alpha$.
3. One may use this property to scale "eigenvalues" to be either 0 or 1 in the complex case, and also in the real case when the degree p is even; for the real case with odd p one may achieve "eigenvalue" 0, 1 or -1 by scaling.
4. Alternatively, one may prescribe that the (Euclidean) norm of an eigenvector should be equal to 1; then the notion of eigenvalue becomes relevant. This will be done later for tensors and their gradient systems.

We now list a number of results on eigenspaces of homogeneous polynomial maps; most of these are based on the work of H. Rohrl [22–26].

Theorem 2. *Let B be as in (5), and $\mathbf{K} = \mathbf{C}$. Then the following hold.*

(i) The number of one-dimensional eigenspaces of B is either infinite or equal to
$$N_R = \frac{p^q - 1}{p - 1},$$
counting multiplicities.

(ii) Test for multiplicity one: Let $v \in \mathbf{C}^q$ be nonzero and $B(v) = \alpha v$ with some $\alpha \neq 0$. Then $\mathbf{C}v$ corresponds to a solution of multiplicity one if and only if the Jacobian $DB(v)$ does not admit the eigenvalue α.

(iii) If the equation $B(x) = 0$ has only the trivial solution $x = 0$ then the number of one-dimensional eigenspaces of B is finite.

[3] This is of course the same situation met in the Fundamental Theorem of Algebra: we know that a polynomial of degree n in one variable always has n roots (counting multiplicities) but, with no further study, we do not know how many of these are real.

(iv) *There is an open and dense subset of coefficient space such that every B with coefficients in this subset admits a basis for $\mathbf{C}^q$ of eigenvectors.*
(v) *There is an open and dense subset of coefficient space such that every B with coefficients in this subset admits exactly N_R different one-dimensional eigenspaces.*
(vi) *If the coefficients $B^i_{j_1\dots j_q}$ are algebraically independent over the rational number field then the equation $B(x) = 0$ has only the trivial solution, and B admits exactly N_R different one-dimensional eigenspaces.*

Proof. We just sketch some arguments for the proofs, and give references. (See also the review [31].) The first assertion is due to Rohrl [22, 24], the second is derived from the familiar criterion for multiplicity one (i.e. invertibility of the linearization). The third assertion is shown e.g. in [30], based on the fact that any projective variety of positive dimension intersects every hyperplane. The fourth and fifth assertion go essentially back to Rohrl [25], although the full statement given in this paper is not correct, and the proof has to be modified. See the Appendix of [14] for a full discussion. The last statement is again due to Rohrl [22]; the algebraic independence condition guarantees that the multiplicity one criterion is always satisfied. □

We turn to the real setting. For proofs and references concerning the following statements we refer to [31]. (In some of the proofs analytic techniques enter the picture.)

Theorem 3. *Let B be as in (5), and $\mathbf{K} = \mathbf{R}$. Then the following hold.*
(i) *If the dimension q is odd then there exists a one-dimensional real eigenspace of B.*
(ii) *If the dimension q is even and the degree p of B is even then there exists a one-dimensional real eigenspace of B.*
(iii) *If the complexification admits finitely many one-dimensional eigenspaces, then the number of real eigenspaces is congruent to N_R modulo 2.*

3.2. **Radial solutions of fully nonlinear dynamical systems.** Rohrl was interested in (one-dimensional) eigenspaces of homogeneous polynomial maps because they give rise to special solutions of an associated differential equation, similar to the linear case. Rohrl considered first order differential equations, and we paraphrase his result here (see the original work in [22] and [23]).

Theorem 4. *Let B be as in (5), and consider the ordinary differential equation*

$$\dot{x} = B(x)$$

in $\mathbf{K}^n$. Then every one-dimensional eigenspace of B is an invariant set for this differential equation.

For nonzero v with $B(v) = \alpha v$, some $\alpha \in \mathbf{K}$, one obtains solutions with the ansatz $x(t) = \xi(t) \cdot v$, which leads to the one-dimensional equation $\dot{\xi} = \alpha \xi^p$.

There is a straightforward extension of this approach to second order equations.

Theorem 5. *Let B be as in (5), and consider the second order ordinary differential equation in $\mathbf{K}^n$*

$$\ddot{x} \;=\; B(x)\ .$$

(i) *Then every nonzero v with $B(v) = \alpha v$, some $\alpha \in \mathbf{K}$, gives rise to special solutions of the differential equation, via the ansatz $x(t) = \gamma(t) \cdot v$, which leads to the one-dimensional second order equation $\ddot{\gamma} = \alpha \gamma^p$.*

(ii) *In case $\mathbf{K} = \mathbf{R}$ a phase plane analysis of the associated system*

$$\dot{y}^1 = y^2,\ \dot{y}^2 = \alpha \left(y^1\right)^p$$

yields the first integral $\psi = 2\alpha \left(y^1\right)^{p+1} - (p+1) \left(y^2\right)^2$. When $\alpha \neq 0$ then the level sets of ψ are bounded if and only if $\alpha < 0$ and p is odd. In this case, the level sets are orbits of periodic solutions of the second order system; in all other cases every nonconstant solution obtained by the ansatz is unbounded.

Proof. (Sketch). Note that 0 is the only stationary point of the system. It is elementary to see that the level sets are bounded, hence compact, if and only if $\alpha < 0$ and p is odd. By standard Poincaré-Bendixson theory for planar systems, the only possible limit sets of points on a compact level set containing more than one point are closed orbits, thus they must coincide with the level sets by connectedness. In every other case, each level set that contains more than one point is unbounded, and any α or ω limit point of a solution starting on such a level set consists of a stationary point, by Poincaré-Bendixson. Unboundedness of the solution follows. □

Remark 1. It is worth noting that in the case $p = 2$ the special solutions from the theorem correspond to a well-known class of special functions. Indeed, the second order equation

$$\ddot{z} \ = \ \alpha \ z^2$$

becomes, upon employing the first integral,

$$\dot{z}^2 \ = \ \frac{2}{3} \ \alpha \ z^3 \ + \ c$$

with some constant c, and this is the differential equation for a Weierstrass $\wp$-function. So, elliptic functions appear in a natural manner.

3.3. **Critical points on the unit sphere.** In this section we are only interested in the real case $\mathbf{K} = \mathbf{R}$. We consider a symmetric tensor $T_{i_1 \dots i_n}$ of rder n on $\mathbf{R}^m$, and we associate with this a polynomial

$$P_n(x) \ := \ T_{i_1 \dots i_n} x^{i_1} \dots x^{i_n} \ ;$$

note that here the dimension m of the ambient space and the degree n of the polynomial are not related. In the following, P_n will also be called the *potential*; it will also be just denoted as P, when we do not need to emphasize its degree. Conversely, as is well known, the algebra of homogeneous polynomials of degree n in $\mathbf{R}^m$ is isomorphic to the algebra of symmetric tensors of the same order n over $\mathbf{R}^m$.

Consider now the gradient of P_n, i.e. the m-dimensional vector

$$\nabla P_n \ = \ \left(\frac{\partial P_n}{\partial x^1} \ , \ \dots \ , \ \frac{\partial P_n}{\partial x^m} \right) \ ;$$

here of course each component is a homogeneous function of degree $(n-1)$ in the x^i.

We define an *eigenvector of the tensor T* to be an eigenvector of (∇P_n).

For eigenvectors of tensors we obtain an improvement of earlier results concerning the real case. We collect them in the following.

Theorem 6. *Let $v \in \mathbf{R}^m$ be of Euclidean norm one. Then*

(i) v is an eigenvector of ∇P_n if and only if v is a critical point of P_n on the unit sphere $S^{m-1} \subset \mathbf{R}^m$.

(ii) The real homogeneous gradient map ∇P_n admits a real eigenvector.

(iii) If we have finitely many critical points x_k, then the sum of the indices of all critical points is equal to the (Euler-Poincaré) characteristic $\chi(S^{m-1})$ of the ambient sphere, thus:

$$\sum_k \iota(x_k) \;=\; \chi(S^{m-1}) \;=\; \begin{cases} 2 & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even} \end{cases} \tag{7}$$

Proof. To prove the first assertion (i), introduce a Lagrange multiplier λ and consider the modified potential

$$\widehat{P}(\mathbf{x}) \;:=\; P(\mathbf{x}) \;-\; \frac{1}{2}\,\lambda\,|\mathbf{x}|^2\;. \tag{8}$$

The gradient of $\widehat{P}$ is given by

$$\nabla\widehat{P} \;=\; \nabla P \;-\; \lambda\,\mathbf{x}\;;$$

hence the solutions to $\nabla\widehat{P} = 0$ are exactly the points on the unit sphere such that $(\nabla P)(\mathbf{x}_0)$ is collinear to $\mathbf{x}_0$; these identify (unit length) eigenvectors of T and hence eigenspaces. The second assertion (ii) is then clear since P_n attains maximum and minimum on the compact unit sphere. The notion of *index* of a critical point a is defined in [16] via the Brouwer degree. When a is nondegenerate (i.e., the derivative at a is invertible) it is equal to the sign of its Jacobian determinant. For the proof of the last assertion (iii) see [2, 16]. □

In view of the results above we will adopt the convention that whenever reference is made to eigenvalues associated with eigenvectors, it is understood that these are associated with eigenvectors of unit length.[4]

We also note that (like for matrices) if T depends on parameters then the eigenvectors and eigenvalues will in general depend on these parameters. We anticipate that also the *number* of independent eigenvectors (that is, eigenspaces) can vary depending on such parameters. This situation is met already in the simplest nontrivial case, i.e. for completely symmetric cubic tensors in three-dimensional space[5]. For a full discussion of this case we refer to [12] (with a more physical approach) and especially to [10].

The situation is specially simple when the ambient space is $\mathbf{R}^3$ (a case of clear physical interest!) and hence we work in $S^2 \subset \mathbf{R}^3$ and all critical points are nondegenerate. In this case maxima and minima have index +1 while saddle points have index -1.

[4] Note that for n odd this still leaves an ambiguity, as the eigenvalues for v and $-v$ differ by sign; it would actually be convenient to consider these as two distinct eigenvalues, corresponding to distinct eigenvectors, in view of the discussion in Sect. 3.3, see [10, 12].

[5] For completely symmetric cubic tensors in two-dimensional space we get a degenerate situation; see the discussion in [28]. We will see later on, in Section 4, that a similar degeneration is met for quartic tensors in two-dimensional space.

Max	*Min*	S_1	S_2	S_3	*NCP*
1	*1*	*0*	*0*	*0*	*2*
2	*2*	*2*	*0*	*0*	*6*
3	*3*	*4*	*0*	*0*	*10*
3	*3*	*0*	*2*	*0*	*8*
4	*4*	*6*	*0*	*0*	*14*
4	*4*	*2*	*2*	*0*	*12*
4	*4*	*0*	*0*	*2*	*10*

TABLE 1. Different possibilities for the number and type of critical points in the case of a cubic potential in three dimensions; here "Max"and "Min" represent the number of maxima and minima, while "S_k" represents the number of saddle points of index $-k$. Finally, "NCP" is the total number of critical points.

In any case, it is elementary to classify all possible combinations of non-degenerate critical points compatible with the formula (7) *and* with Bezout's theorem, which provides the maximal number of real critical points[6]. In view of the special nature of the potential considered here (homogeneous of degree n), it is either even or odd, depending on the parity of n, hence all critical points are dual to each other under reflection. Thus in the odd case there will be as many maxima as minima while in the even one these numbers will necessarily be even, as well as (in all cases) the number of saddles of any given index.

For example, in [10] it is argued that in the case of a cubic potential (and hence a quadratic gradient mapping) in three-dimensional space (and hence a two-dimensional ambient sphere) with non-degenerate extremals (but possibly degenerate saddles) only the possibilities listed in Table 1 arise.[7]

Here we are more interested in even degree, and especially in *quartic* potentials, and hence cubic gradient mappings. In this case, already for ambient space $\mathbf{R}^3$ we get up to $(3^3 - 1) = 26$ critical points and a complete classification would make little sense. We remark, however, that Table 1 is still valid in that it concerns topological features; on the other hand, in this context it does *not* provide a complete classification of the possible situations, but only of those with no more than four maxima or minima.

We also note that when the ambient space is $\mathbf{R}^2$, and hence the relevant sphere is just a circle S^1, then we always have as many maxima as minima, whose number is of course limited by the degree of the potential; and of course no saddles. E.g. for $p = 3$ and $q = 2$ we have at most four maxima and four minima.

Some simple Examples will be considered in detail in Section 4.

4. **Example. Invariant lines; existence and number of higher order normal modes.** As the simplest possible example of the situation we have been studying, we consider a point particle of mass $m = 1$ in $\mathbf{R}^2$ (with cartesian coordinates

[6]Note that while the characteristic is an intrinsic property of the sphere we work on, the bound provided by Bezout's and by Rohrl's theorems depends on the degree of the mapping.

[7]It should be noted that not all of these are realized when we consider, as the Physics of the problem studied in [10, 12] requires, *completely traceless* tensors. See also the discussion in [31] in this respect.

x, y) evolving under the action of a quartic potential (a cubic one would not satisfy the requirement that the origin is a stable equilibrium); in order to reduce the complexity of the potential (and hence of the analysis) we assume it depends on x and y only through their squares. That is,

$$V(x,y) \;=\; a\,x^4 \;+\; b\,y^4 \;+\; 2\,c\,x^2\,y^2 \;. \tag{9}$$

Note that this is symmetric under $Z_2 \times Z_2$ (these acting as reflections in x and in y); if we wish to require this to be also invariant under the Z_2 involution exchanging x and y then we should require $b = a$. In this case it would be convenient, with a suitable redefinition of constants, to rewrite this as

$$V(x,y) \;=\; \alpha\;(x^2+y^2)^2 \;+\; \beta\,x^2\,y^2 \;. \tag{10}$$

We will refer to these cases as the *lower symmetry* and the *higher symmetry* cases respectively.

4.1. **The higher symmetry case.** We will first consider the case where the system is $Z_2 \times Z_2 \times Z_2$ symmetric, i.e. the potential is in the form (10).

Note that in order to have a stable point at the origin, the coefficients α should be positive; we will assume this to be the case from now on.

Moreover, we can always rescale V (which amounts to a rescaling of time) and choose $\alpha = 1$. Then in order to have a minimum at the origin we can ask $\beta > -4$; this is obtained by looking at the behavior of the potential $V_m(x) := V(x, mx)$ along all lines $y = mx$ (including the $m = \infty$ case, i.e. the y axis). We will moreover assume $\beta \neq 0$ to avoid the fully degenerate case with rotational symmetry.

When we pass to polar coordinates (we consider $\theta \in (-\pi, \pi]$, and of course $\rho \in [0, \infty)$)

$$x \;=\; \rho\;\cos(\theta)\;, \quad y \;=\; \rho\;\sin(\theta)$$

and constrain the potential on the unit circle $\rho^2 = x^2 + y^2 = 1$, call it $W(\theta)$, we get

$$W \;=\frac{1}{8}\;[8 \;-\; \beta\,\cos(4\,\theta) \;+\; \beta] \;. \tag{11}$$

Thus we have

$$\frac{dW}{d\theta} \;=\; \frac{1}{2}\;\beta\;\sin(4\,\theta)\;;$$

critical points are obtained for

$$\theta \;=\; \theta_k \;:=\; k\;\frac{\pi}{4}\;, \quad |k| \leq 4\;.$$

That is, we get *eight* critical points on the unit circle, corresponding to *four* invariant lines; this applies for *any* nonzero value of β (as already remarked $\beta = 0$ is the fully rotationally invariant and hence infinitely degenerate case; we excluded this from our considerations).

The stability of the critical points is controlled by

$$\left[\frac{d^2W}{d\theta^2}\right]_{\theta_k} \;=\; 2\;\beta\;\cos(4\,\theta_k)\;.$$

Thus we have a bifurcation at $\beta = 0$; for this value of β all the stabilities are exchanged. In particular, for $\beta < 0$ the lines identified by $\theta = \pm\pi/4$ are stable and the axes ($\theta = 0, \pi/2$) are unstable, while for $\beta > 0$ the lines identified by $\theta = \pm\pi/4$ are unstable and the axes are stable; see Figure 1.[8]

[8] We stress that here the stability means stability for the potential restricted to the unit circle, while the origin is *always* stable for the full two-dimensional potential $V(x, y)$.

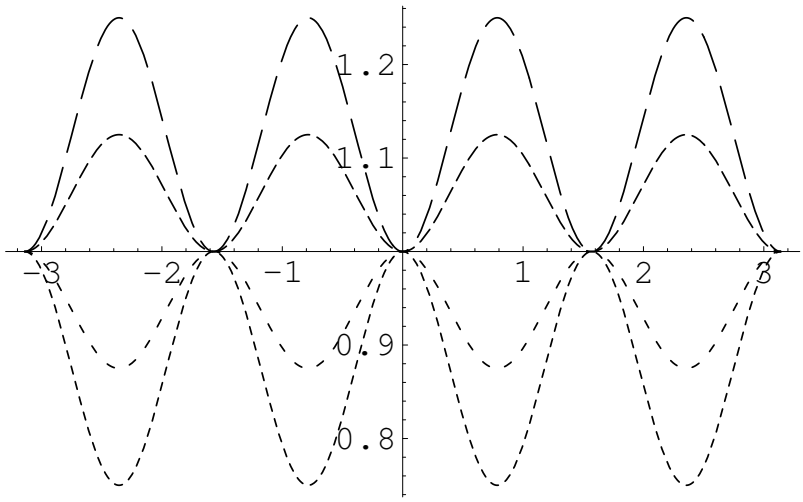

FIGURE 1. The potential $W(\theta)$ as in eq.(11) for different values of β; here $\beta = -1, -0.5, 0.5, 1$. The exchange of stability takes place at $\beta = 0$.

The effective potentials along the invariant lines, call these as V_θ where θ is the (invariant) angular coordinate in the (x, y) plane, is always of the form $V_\theta(r) = c_\theta\, r^4$ with c_θ a constant. We actually get (recall we assumed $\beta > -4$)

$$c_0 \;=\; c_{\pi/2} \;=\; 1\;;\;\; c_{\pi/4} \;=\; c_{-\pi/4} \;=\; 1 \;+\; \beta/4\;.$$

This is coherent, of course, with the stability of the equilibrium point at the origin.

We stress that, apart from the stability exchange for $W(\theta)$, in this case there are no qualitative changes as the parameters (which in this case means just the parameter β) are varied: we always have four critical lines and hence four normal modes; two of them are stable and two of them unstable.

Some numerical simulations of this dynamics for initial data near to the (higher order) normal modes are shown in Figure 2; they confirm stability as discussed above.[9]

4.2. **The lower symmetry case.** We will now consider the general form (which has only $Z_2 \times Z_2$ as symmetry), i.e. the potential (9). With no loss of generality, we can assume $a > b$ (if not, just switch x and y). Here again stability requires that both a and b are positive; by a rescaling we can set $a = 1$, and hence $0 < b < 1$, and deal with the potential

$$V(x, y) \;=\; x^4 \;+\; \alpha\, y^4 \;+\; 2\,\beta\, x^2\, y^2\;, \tag{12}$$

where $0 < \alpha < 1$.

By looking at this along the line $y = mx$ we get

$$V_m(x) \;:=\; V(x, mx) \;=\; (1 + 2\beta m^2 + \alpha m^4)\; x^4\;;$$

stability of the origin requires therefore that

$$1 \;+\; 2\,\beta\, m^2 \;+\; \alpha\, m^4 \;>\; 0$$

[9]Here we show the situation for $\beta > 0$; simulations for $\beta < 0$ would also confirm our discussion, and are not shown for the sake of brevity.

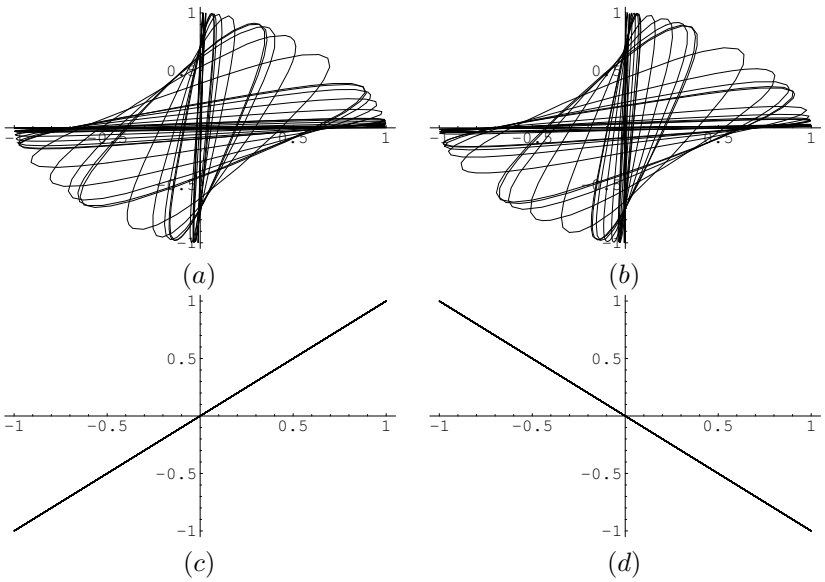

FIGURE 2. Numerical integration of the motion generated by the potential (10) with the choice $\beta = 1$ for initial conditions near to normal modes. In all cases, initial data correspond to zero speed and position at $r = 1$ along eigenvectors, with an offset of 0.001 from the latter. The simulation show the outcome, for $t \in (0, 100)$, for initial data: (a) near the eigenvector $\theta = 0$, (b) near the eigenvector $\theta = \pi$, (c) near the eigenvector $\theta = \pi/4$, (d) near the eigenvector $\theta = -\pi/4$.

for all choices of m, and this implies

$$\beta \; > \; - \sqrt{\alpha} \; ,$$

which we assume from now on.

After passing to polar coordinates, the restriction of V given by (12) to the unit sphere reads

$$W(\theta) \; = \; \cos^4(\theta) \; + \; \alpha \; \sin^4(\theta) \; + \; 2\,\beta \, \sin^2(\theta) \cos^2(\theta) \; , \tag{13}$$

and from this we get at once

$$\frac{dW}{d\theta} \; = \; - \; [1 \, - \, \alpha \, + \, (1 + \alpha - 2\beta) \, \cos(2\theta)] \; \sin(2\theta) \; .$$

Thus critical points of W are identified either by $\sin(2\theta) = 0$, i.e. by $\theta = 0, \pm\pi/2, \pm\pi$; or by

$$\cos(2\theta) \; = \; \frac{\alpha - 1}{\alpha + 1 - 2\beta} \; .$$

Solutions to this equation exist only in the regions $\beta < \alpha$ and $\beta > 1$ (recall that we assumed $0 < \alpha < 1$), but not for $\alpha < \beta < 1$.

Thus we conclude that β lies in the range $\beta > -\sqrt{\alpha}$, and that for β taking the values $\beta = \alpha$ and $\beta = 1$ there are bifurcations changing the number of critical points for W, i.e. of invariant lines for our potential $V(x, y)$.

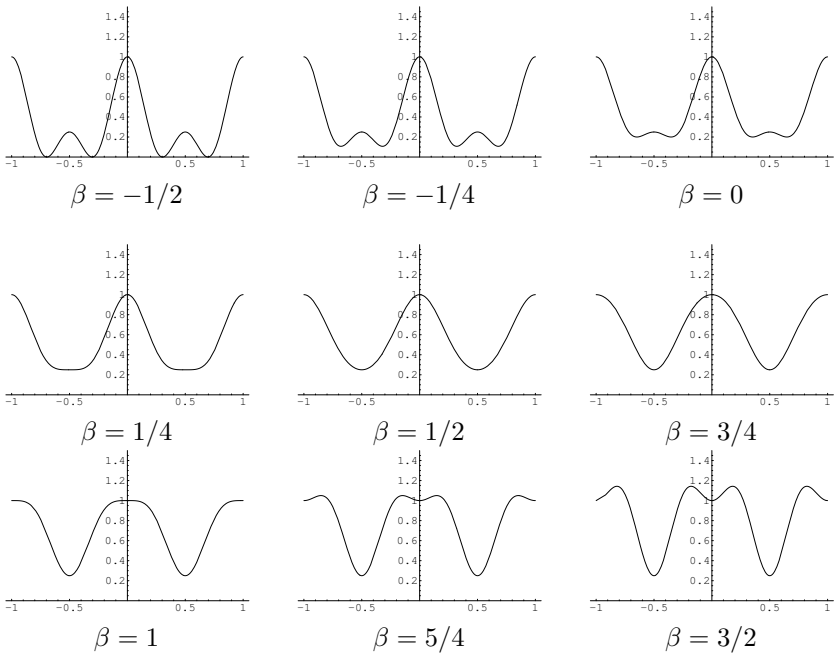

FIGURE 3. The potential $W(\theta)$, see (13), for $\alpha = 1/4$ and various choices of β. Here θ is measured in units of π.

More precisely, we always have four critical points at $\theta = 0, \pm\pi/2, \pi$; moreover for suitable β we also have four more critical points at

$$\theta \;=\; \pm\,\frac{1}{2}\,\arccos\left[\frac{1-\alpha}{1+\alpha-2\beta}\right] \;:=\; \pm\theta_*\;.$$

That is, we have either eight or four critical points for the potential W restricted on the sphere, corresponding to *four* or *two* critical lines and hence normal modes, depending on the value of β. See Figure 3.

The stability of critical points is controlled by

$$\frac{d^2W}{d\theta^2} \;=\; -\,2\;\left[(1-\alpha)\;\cos(2\,\theta)\;+\;(1+\alpha-2\beta)\;\cos(4\theta)\right]\;;$$

in particular at the various critical points identified above we have

$$\begin{aligned}
(d^2/d\theta^2)_{\theta=0} \;&=\; (d^2/d\theta^2)_{\theta=\pi} \;=\; -\,4\;(1-\beta)\;;\\
(d^2/d\theta^2)_{\theta=\pi/2} \;&=\; (d^2/d\theta^2)_{\theta=-\pi/2} \;=\; -\,4\;(\alpha-\beta)\;;\\
(d^2/d\theta^2)_{\theta=\theta_*} \;&=\; (d^2/d\theta^2)_{\theta=-\theta_*} \;=\; 8\;\left[\frac{(\alpha-\beta)\;(1-\beta)}{1+\alpha-2\beta}\right]\;.
\end{aligned}$$

This immediately shows that the critical line corresponding to $\theta = 0$ (equivalently, to $\theta = \pi$) undergoes a change of stability at $\beta = 1$; and the critical line corresponding to $\theta = \pi/4$ (equivalently, to $\theta = 3\pi/4$) undergoes a change of stability at $\beta = \alpha$. As for the critical lines corresponding to $\theta = \pm\theta_*$, existing in the regions $\beta < \alpha$ and $\beta > 1$, noting that $\alpha < (1+\alpha)/2 < 1$, we have that these are stable for $\beta < \alpha$ and unstable for $\beta > 1$. (Numerical simulations, not shown for the sake of brevity, confirm again our analysis.)

5. **Discussion and conclusions.** In this note we have considered natural Hamiltonians for point particle, $H = K + V$, for which the dynamics in local coordinates is simply $\ddot{x} = -\nabla V$. We have considered the neighborhood of a stable equilibrium x_0, and studied the case where the Taylor expansion of V starts with terms of order k higher than two (we have of course given special attention to the case $k = 4$).

We have discussed how the notion of *normal modes* is modified in this case; this is based essentially on known results concerning *eigenvalues of (homogeneous) tensors*. It results that higher order normal modes do exist, but while some of their properties extend from the standard (i.e. $k = 2$) case to the present one, other do not. In particular, their number is not fixed and can exceed the dimensionality of the ambient space; moreover the most general dynamics near the equilibrium is in general[10] *not* a superposition of normal modes, at difference with the standard case.

It should be mentioned that a question which arise naturally has not been studied here, and should be considered in the future. This is of course the *persistence* of these higher order normal modes under perturbations, i.e. when one considers also higher order terms in the series expansion for the potential around the equilibrium point. We recall that for standard normal modes a theory of persistence exists [17, 29]; this is based on variational analysis and guarantees persistence of some of the normal nodes under certain conditions. Apart from an extension to the new higher order normal modes along this line of attack, one could also consider an approach based on the theory of Poincaré-Birkhoff *normal forms* [3, 9] (or some generalization thereof).

It is worth noting that for the examples discussed in Section 4, persistence of some modes can be guaranteed by symmetry arguments if the symmetry is preserved by higher order perturbations: In both cases, the potential then admits symmetries sending $x \mapsto x$, $y \mapsto -y$, resp. $x \mapsto -x$, $y \mapsto y$ (analogously for the time derivatives), hence the fixed point spaces of these symmetries are necessarily invariant. The fixed point spaces correspond to the cases $\theta = 0, \pm\pi$ above. In the higher symmetry case one also has symmetries exchanging x and y, resp. x and $-y$, with invariant fixed point spaces corresponding to $\theta = \pi/4$ resp. $\theta = 3\pi/4$.

We plan to tackle the persistence problem generally in a later publication.

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[10] It is conceivable that one can build systems for which a *nonlinear superposition principle* [6,7] holds. We did not discuss this point here.

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