Symmetry of stochastic differential equations

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Symmetry & equations

- The modern theory of Symmetry was laid down by Sophus Lie (1842-1899).
- The motivation behind the work of Lie was not in pure algebra, but instead in the effort to *solve differential equations*.
- This was successful !
- Can we do something similar for *stochastic* differential equations ?

This talk

- I first illustrate how the theory of symmetry helps in determining solutions of (deterministic) differential equations, both ODEs and PDEs

- I will be staying within the classical theory (Lie-point symmetries), work in coordinates, and only consider continuous symmetries.

- I will then discuss the extension of this theory to *stochastic (ordinary) differential equations*.

This talk

An important topic will be absent from my discussion: *symmetry of variational problems* (Noether theory)

Two good reasons for this (beside the shortage of time):

$\diamondsuit$ everybody here is familiar with this theory in the deterministic framework;

$\diamondsuit$ I am not familiar with this theory in the stochastic framework.

Symmetry of deterministic equations

The Jet space

Key idea (Cartan, Ehresmann): introduce the *jet bundle* (here jet space).

Phase space (bundle): space of dependent ($u^1, ..., u^p$) and independent ($x^1, ..., x^q$) variables; (M, π_0, B).
Jet space (bundle): space of dependent ($u^1, ..., u^p$) and independent ($x^1, ..., x^q$) variables, together with the partial derivatives (up to order n) of the u with respect to the x; $(J^n M, \pi_n, B)$.

Geometry of differential equations

A differential equation Δ determines a manifold in $J^n M$, the *solution manifold* $S_\Delta \subset J^n M$ for Δ.

This is a geometrical object, the differential equation can be identified with it, and we can apply geometrical tools to study it.

How to keep into account that u^a_J represents derivatives of the u^a w.r.t. the x^i ?

The jet space should be equipped with an additional structure, the *contact structure*.

Contact structure

This can be expressed by introducing the one-forms

$$\omega^a_J \ := \ \mathrm{d}u^a_J \ - \ \sum_{i=1}^q u^a_{J,i} \, \mathrm{d}x^i$$

(contact forms) and looking at their kernel.

Prolongation

An infinitesimal transformation of the x and u variables is described by a *vector field* in M; once this is defined the transformations of the derivatives are also implicitly defined.

The procedure of extending a VF in M to a VF in $J^n M$ by requiring the preservation of the contact structure is also called *prolongation*.

Symmetry. 1

A VF X defined in M is then a *symmetry* of Δ if its prolongation $X^{(n)}$, satisfies

$$X^{(n)} \ : \ S_\Delta \ \to \ \mathrm{T} S_\Delta \ .$$

An equivalent characterization of symmetries is to map solutions into (generally, different) solutions.

In the case a solution is mapped into itself, we speak of an *invariant solution*.

Symmetry. 2

A first use of symmetry can be that of *generating new solutions from known ones*.

Example: the solution $u = 0$ *to the heat equation get transformed by symmetries into the fundamental (Gauss) solution.*

This is not the only way in which knowing the symmetry of a differential equation can help in determining (all or some of) its solutions.

Determining the symmetry of a differential equation

Determining the symmetry of a given differential equation goes through solution of a system of coupled *linear* PDEs.

The procedure is algorithmic and can be implemented via computer algebra...

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(Except for first order ODEs !)

Using the symmetry

The key idea is the same for ODEs and PDEs, and amounts to the use of *symmetry adapted coordinates* (XIX century math!)

But the scope of the application of symmetry methods is rather different in the two cases.

We will consider scalar equations for ease of discussion

Symmetry and ODEs. 1

If an ODE Δ of order n admits a Lie-point symmetry, the equation can be *reduced* to an equation of order $n-1$.

The solutions to the original and to the reduced equations are in correspondence through a quadrature (which of course introduces an integration constant).

Symmetry and ODEs. 2

The main idea is to change variables

$$(x, u) \to (y, v) \ ,$$

so that in the new variables

$$X = \frac{\partial}{\partial v} \ .$$

As X is still a symmetry, this means that the equation will not depend on v, only on its derivatives.

With a new change of coordinates

$$w := v_y$$

we reduce the equation to one of lower order.

Symmetry and ODEs. 3

A solution $w = h(y)$ to the reduced equation identifies solutions $v = g(y)$ to the original equation (in "intermediate" coordinates) simply by integrating,

$$v(y) \ = \ \int w(y)\, dy \ ;$$

a constant of integration will appear here.

Finally go back to the original coordinates inverting the first change of coordinates.

Symmetry and ODEs. 4

The reduced equation could still be too hard to solve;

The method can only guarantee that we are reduced to a problem of lower order, i.e. hopefully simpler than the original one.

Solutions to the original and the reduced problem are in correspondence

Symmetry and PDEs. 1

The approach in the case of PDEs is in a way at the opposite as the one for ODEs!

If X is a symmetry for Δ, change coordinates

$$(x, t; u) \ \to \ (y, s; v)$$

so that in the new coordinates

$$X \ = \ \frac{\partial}{\partial y} \ .$$

Now our goal will *not* be to obtain a general reduction of the equation, but instead to obtain a (reduced) equation which determines the *invariant solutions* to the original equation.

Symmetry and PDEs. 2

In the new coordinates, this is just obtained by *imposing* $v_y = 0$, i.e. $v = v(s)$.

The reduced equation will have (one) less independent variables than the original one.

This reduced equation will *not* have solutions in correspondence with general solutions to the original equation: only the invariant solutions will be common to the two equations

Contrary to the ODE case, we do not need to solve any "reconstruction problem".

Symmetry and linearization

It was shown by Bluman and Kumei that the (algorithmic) symmetry analysis is also able to detect if a nonlinear equation can be linearized by a change of coordinates.

The reason is that underlying linearity will show up through a Lie algebra reflecting the *superposition principle*.

Generalized symmetries

The concept of symmetry was generalized in many ways.

This extends the range of applicability of the theory

We have no time to discuss these.

Symmetry of stochastic vs. diffusion equations

Symmetry and SDEs. 1

Consider SDEs in Ito form,

$$dx^i \ = \ f^i(x,t)\,dt \ + \ \sigma^i{}_k(x,t)\,dw^k \ ;$$

I will only consider *ordinary* SDEs.

- Here again I will *not* consider variational problems.

Symmetry and SDEs. 2

The first attempts to use symmetry in the context of SDEs involved quite strong requirements for a map to be considered a symmetry of the SDE.

They were based on the idea of a symmetry as a map taking solutions into solutions.

The first approach required that for any given realization of the Wiener process any sample path satisfying the equation would be mapped to another such sample path.

It is not surprising that the presence of symmetries was then basically related to situations where, in suitable coordinates, the evolution of some of the coordinates was deterministic and not stochastic.

Symmetry and SDEs. 3

A step forward in considering symmetry for SDEs independently from a variational origin was done when an Ito equation was associated to the corresponding diffusion equation.

The idea behind this is that a sample path should be mapped into an *equivalent* one.
(Here equivalence is meant in statistical sense.)

Symmetry and SDEs. 4

We thus have two types of symmetries for the one-particle process described by a SDE: the equation can be invariant under the map, or it may be mapped into a *different* equation which has the *same* associated diffusion equation.

In this way one is to a large extent considering the symmetries of the associated FP equation, and this had been studied in detail in the literature.

Symmetries of the Ito equation are also symmetries for the FP, while the converse is not necessarily true.

Symmetry and SDEs. 5

The theory can be extended to consider also transformations acting on the Wiener processes (W-symmetries)

And to consider random dynamical systems (RDS) defined by an Ito equation beside the one particle process (OPP) defined by the same Ito equation.

Not surprisingly, it turns out that – for a given Ito equation – any symmetry of the associated RDS is also a symmetry of the OPP, while the converse is not true.

More recently the “diffusive” approach to symmetries of SDEs has been reconsidered by F. De Vecchi in his (M.Sc.) thesis, making contact with so called “second order geometry” developed by Meyer and Schwartz.

Symmetry and SDEs. 6

The works mentioned so far focused on determining what is the “right definition” for symmetries of a SDE.

If we have a symmetry, we should use (as in the deterministic case, and is done in stochastic Noether theory) *symmetry-adapted coordinates*.

This was undertaken by Meleshko and coworkers, and they promptly showed that using these coordinates leads to many advantages in concretely dealing with SDEs. (Work in this direction is also being pursued by De Vecchi and Ugolini in Milano.)

Symmetry and SDEs. 6

They determined e.g. conditions for the linearization of SDEs in terms of symmetry.

Moreover, using this linearization approach, they studied how symmetries can be used to integrate a SDE.

These are concerned with the most favorable case; in the deterministic case one is in general not that ambitious.

One would expect that such a less optimistic approach would be of use also in the case of SDEs.

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Symmetry of SDEs

Types of symmetries for SDEs. 1

(work with Francesco Spadaro, Roma, now in EPFL)

$$
\begin{aligned}
dx^i &= f^i(x,t)\,dt \;+\; \sigma^i_j(x,t)\,dw^j \\
X &= \tau\,\partial_t \;+\; \xi^i\,\partial_i
\end{aligned}
$$

- Simple symmetries (act only on the x)
- General symmetries (act on both the x and t)
- W-symmetries (act also on the w^j)

Types of symmetries for SDEs. 2

(work with Francesco Spadaro, Roma, now in EPFL)

$$\begin{aligned} dx^i &= f^i(x,t)\,dt \;+\; \sigma^i_j(x,t)\,dw^j \\ X &= \tau\,\partial_t \;+\; \xi^i\,\partial_i \end{aligned}$$

- Deterministic symmetries: $\xi = \xi(x,t)$, $\tau = \tau(x,t)$
- Random symmetries: $\xi = \xi(x,t,w)$, $\tau = \tau(x,t,w)$
- W symmetries: $X = \tau\partial_t + \xi^i\partial_i + h^k\widehat{\partial}_k$ with $\xi = \xi(x,t,w)$, $\tau = \tau(x,t,w)$, $h^k = B^k_\ell w^\ell$.

Symmetry of SDEs. 1

When we look at symmetry of a SDEs *per se* a substantial problem is present:

- The symmetry approach is based on passing to symmetry-adapted coordinates;
- Vector fields transform “geometrically” (chain rule) upon changes of coordinates
- Deterministic DEs are (identified with) geometrical objects, hence also transform geometrically
- It is then obvious that symmetry are preserved under changes of coordinates

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Symmetry of SDEs. 2

• On the other hand, an Ito equation is NOT a geometrical object

• In fact, it transforms under the Ito rule, not the chain rule

• Thus it is **not** granted that X will still be a symmetry when we change coordinates so that $X = \partial_x$!

• This is also true for *deterministic* symmetries of stochastic equations

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Symmetry of SDEs. 3

The easy way out would be using Stratonovich equations

- These do transform according to the chain rule, i.e. geometrically
- But the relation between an Ito and the corresponding Stratonovich process is not that obvious – especially in this respect
- In fact, it is known that in general the two do not share the same symmetries [Unal]...
- albeit they have the same simple symmetries
- which is interesting, as Kozlov theory relating symmetry to integrability of SDEs only makes use of simple symmetries

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Unal type theorems

Proposition 1 (Unal). *The simple* deterministic *symmetries of an Ito equation and those of the equivalent Stratonovich equation do coincide.*

Proposition 2 (GG+Lunini). *The simple* deterministic or random *symmetries of an Ito equation and those of the equivalent Stratonovich equation do coincide.*

Unal type theorems

Unal also showed that – even in the deterministic framework – the result does *not* extend to more general symmetries; if one considers symmetries with generator

$$X = \tau(\partial/\partial t) + \varphi^i(\partial/\partial x^i)$$

the determining equations for the Ito and the associated Stratonovich equation are equivalent if and only if τ satisfies the additional condition

$$\sigma^k_{\ p}\,\sigma^{ip}\ \left[\partial_k\left(\partial_t\tau\ +\ f^j\,(\partial_j\tau)\ +\ \frac{1}{2}\,\sigma^m_{\ q}\,\sigma^j_{\ q}\,(\partial_m\partial_j\tau)\right)\right]\ =\ 0\ ;$$

This is identically satisfied for $\tau = \tau(t)$ (i.e. for "acceptable" cases according to the discussion in GG+Spadaro).

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Kozlov theory

• In the deterministic case symmetry guarantees an ODE can be reduced (or solved)

• The same holds in the SDE case, but only simple symmetries $X = f^i(x,t)\partial_i$ matter [note now x and t are really different!]

Kozlov first theorem

Theorem 1. *The SDE*

$$dy \;=\; \widetilde{f}(y,t)\,dt \;+\; \widetilde{\sigma}(y,t)\,dw \tag{1}$$

can be transformed by a deterministic map $y = y(x,t)$ *into*

$$dx \;=\; f(t)\,dt \;+\; \sigma(t)\,dw\;, \tag{2}$$

and hence explicitly integrated, if and only if *it admits a simple deterministic symmetry.*

Kozlov first theorem

If the generator of the latter is

$$X = \varphi(y,t)\,\partial_y \ ,$$

then the change of variables $y = F(x,t)$ *transforming (1) into (2) is the inverse to the map* $x = \Phi(y,t)$ *identified by*

$$\Phi(y,t) = \int \frac{1}{\varphi(y,t)}\,dy \ .$$

[The “if” part is due to Kozlov, the “only if” one to C.Lunini]

Kozlov other theorems

The same approach can be pursued to study partial integrability, i.e.reduction of an n-dimensional SDE to an SDE in dimension $n-r$ plus r (stochastic) integrations.

This is possible if and only if there are r simple symmetry generators spanning a solvable Lie algebra.

[Again the “if” part is due to Kozlov, the “only if” one to C.Lunini]

Conclusions

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• The symmetry approach is a general way to attack DEs; in the deterministic framework it proved invaluable both for the theoretical study of differential equations and for obtaining their concrete solutions.

• The theory is comparatively much less advanced in the case of stochastic differential equations.

• There is now some general agreement on what the “right” (that is, useful) definition of symmetry for SDE is.

• but only few applications have been considered, most of these concerning “integrable” equations.

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Conclusions

- Theorems equivalent to the standard ones for ODEs have been obtained for (ordinary) SDEs
- both for what concerns *solving* equations and for *reducing them*
- except that now we cannot use *general* symmetries, but only *simple* ones.

Perspectives

- There is ample space for considering new applications, first and foremost considering “non integrable” equations.

- Correspondingly, there is ample space for concrete applications, i.e. applying the approaches already existing or to be developed to new concrete stochastic systems.

- Symmetry theory flourished and expanded its role by considering generalization of the “standard” (i.e. Lie-point) symmetries in several directions. As far as I know, there is no attempt in this direction for stochastic systems yet; any work in this direction is very likely to collect success and relevant results.

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Perspectives

- So far *only first order systems* have been considered (Einstein-Smoluchowsky *vs.* Ornstein-Uhlenbeck)
- Can we do anything with stochastic formulation of QM ?
- Or at least can we deal directly with symmetries in the Kac-like approach to Wiener (and Ito) processes[a] ?

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[a]This was done some decades ago, but it is not clear how that theory fits with the modern theory of symmetry of DEs

Final word

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Time is now ripe for extending fully fledged symmetry theory to stochastic systems.

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Thank you !

Really Final word

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Happy Birthday

Gianfausto !!!

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